Volume 5, No.4, April 2018
Journal of Global Research in Mathematical Archives
UGC Approved Journal
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

Computing First And Second Zagreb Indices of Generalized Xyz–Point-Line Transformation Graphs

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Abstract: Given a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, let $L(G)$ be the line graph, $\bar{G}$ the complement of G, $S(G)$ the subdivision graph of G and $S^*(G)$ the partial complement of subdivision graph. Let G^0 be the graph with $V(G^0) = V$ and with no edges, G^1 the complete graph with the vertex set V, $G^+ = G$ and $G^- = \bar{G}$. Given $x,\ y,\ z \in \{0, 1, +, -\}$, the *xyz-transformation graph* $T^{xyz}(G)$ of G is the graph with vertex set $V(T^{xyz}(G)) = V \cup E$ and the edge set $E(T^{xyz}(G)) = E(G^x) \cup E(L(G)^y) \cup E(W)$, where $W = S(G)$ if $z = +, W = S^*(G)$ if $z = -, W$ is the graph with $V(W) = V \cup E$ and with no edges if $z = 0$ and W is complete bipartite graph with parts V and E if $z = 1$. This concept was introduced by Deng et al. [9]. Later these *xyz*-transformation graphs are called as *generalized xyz–Point-Line transformation graphs* in [3]. In this paper, we compute the expressions for vertex degree, first and second Zagreb indices of generalized *xyz*–Point-Line transformation graphs.

2010 Mathematics Subject Classification: 05C07.
Keywords: Zagreb indices, Zagreb coindices, generalized *xyz*–Point-Line transformation graph, reformulated Zagreb indices and coindices.

1 Introduction

In this paper, we consider simple and undirected graphs. Here, for all notions on graphs and topological indices that are used but not defined can be found in [7, 8, 14, 15, 19, 30].

Let G be a graph with the vertex set $V(G) = V$ and edge set $E(G) = E$ such that $|V| = n$ and $|E| = m$. As usual n is said to be *order* and m the *size* of G. The *degree of a vertex* $v \in V(G)$ is the number of vertices adjacent to v in G and is denoted by $d_G(v)$. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The *degree of an edge* e in G is denoted by $d_G(e)$, and is defined by $d_G(e) = d_G(u) + d_G(v) - 2$ with $e = uv$. Here $e \sim f$ $(e \nsim f)$ means that the edges e and f are adjacent (resp., not adjacent) and also $u \sim e$ $(u \nsim e)$ means that the vertex u and edge e are incident (resp., not incident) in G.
The *complement* $\bar{G}$ of a graph G is the graph with the same vertex set as G, in which two vertices are adjacent if and only if they are not adjacent in G.
The *line graph* $L(G)$ of a graph G is the graph with vertex set $E(G)$ and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent.
For a graph $G = (V, E)$, let G^0 be the graph with the vertex set $V(G^0) = V$ and with no edges, G^1 the complete graph with $V(G^1) = V$, $G^+ = G$, and $G^- = \bar{G}$. Let $S(G)$ $(S^*(G))$

be the graph with the vertex set $V \cup E$ such that two vertices of $S(G)$ $(S^*(G))$ are adjacent if and only if one corresponds to a vertex v of G and other to an edge e of G and v is incident (resp., not incident) to e in G. Here $S(G)$ is the *subdivision graph* of G and $S^*(G)$ is the *partial complement of subdivision graph*.

Definition[9]. Given a graph $G=(V,E)$ and three variables $x,y,z \in \{0,1,+,-\}$, the *xyz − transformation* $T^{xyz}(G)$ (or T^{xyz}) of G is the graph with the vertex set $V(T^{xyz}) = V \cup E$ and the edge set $E(T^{xyz}) = E(G^x) \cup E(L(G)^y) \cup E(W)$, where $W = S(G)$ if $z=+$, $W = S^*(G)$ if $z=-$, W is graph with $V(W)=V \cup E$ and with no edges if $z=0$, and W is the complete bipartite graph with parts V and E if $z=1$.

Since there are sixty four distinct 3-permutations of $\{1,0,+,-\}$, there are *sixty four* different $xyz-$transformations of a given graph G. These *xyz − transformation graphs* are also called as *generalized xyz − Point-Line transformation graphs*[3]. The vertex v of T^{xyz} corresponding to a vertex v of G is referred to as *point-vertex* and vertex e of T^{xyz} corresponding to an edge e of G is referred to as *line-vertex*[3]. In Figure 1 some self-explanatory examples of T^{xyz} graphs are depicted, dark circles represents the *point-vertices* and light circles represents the *line-vertices* of T^{xyz}.

Theorem 1.1 *[3] Let G be a graph of order n and size m. Then*
$|V(T^{xyz}(G))| = n+m$.
$|E(T^{xyz}(G))| = |E(G^x)| + |E(L(G)^y)| + |E(W)|$,

where $$|E(G^x)| = \begin{cases} 0 & \text{if } x=0. \\ \binom{n}{2} & \text{if } x=1. \\ m & \text{if } x=+. \\ \binom{n}{2}-m & \text{if } x=-. \end{cases}$$

$$|E(L(G)^y)| = \begin{cases} 0 & \text{if } y=0. \\ \binom{m}{2} & \text{if } y=1. \\ -m+\frac{1}{2}M_1 & \text{if } y=+. \\ \binom{m+1}{2}-\frac{1}{2}M_1 & \text{if } y=-. \end{cases}$$

$$|E(W)| = \begin{cases} 0 & \text{if } z=0. \\ mn & \text{if } z=1. \\ m & \text{if } z=+. \\ m(n-2) & \text{if } z=-. \end{cases}$$

Topological indices are numerical parameters of a graph which are invariant under graph isomorphisms. The significance of topological indices is usually associated with quantitative structure property relationship (QSPR) and quantitative structure activity relationship (QSAR) [10, 22, 29]. The idea of topological index appears from work done by Wiener [31] in 1947 although he was working on boiling point of paraffin. He called this index as Wiener index and then theory of topological index started. From these times the number of topological indices has grown enormously, and at present there are more than a thousand topological indices of various complexity [29]. Topological indices were correlated successfully with many physicochemical parameters such as density, viscosity, boiling temperature, solubility, refractive index, molar heat capacity, and standard Gibbs energy of formation [2, 23, 24, 27]. The Zagreb M_1 and M_2 indices were introduced by Gutman and Trinajstic' in 1972 [18]. The first and second Zagreb indices belong among the oldest molecular structure descriptors, and their properties have been extensively investigated [16, 26]. In chemical literature, there have been a few earlier attempts to shift from ordinary molecular graph to their transformations in [4, 5, 17, 20]. This motivated us to compute expressions for first and second Zagreb indices of some generalized $xyz-$Point-Line transformation graphs.

2 Some topological indices

The *first Zagreb and second Zagreb indices* are defined as

$$M_1 = M_1(G) = \sum_{v\in V(G)} d_G(v)^2 \text{ and } M_2 = M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v) \text{ respectively.}$$

The *first Zagreb index* can also expressed as a sum over edges of G,

$$M_1 = \sum_{uv\in E(G)} [d_G(u)+d_G(v)].$$

Do$\breve{s}$lic' [11] defined the *first and second Zagreb coindices* as

$$\overline{M_1} = \overline{M_1}(G) = \sum_{uv\notin E(G)} [d_G(u)+d_G(v)] \text{ and } \overline{M_2} = \overline{M_2}(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v) \text{ respectively.}$$

The vertex-degree-based graph invariant

$$F = F(G) = \sum_{v\in V(G)} d_G(v)^3 = \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2]$$

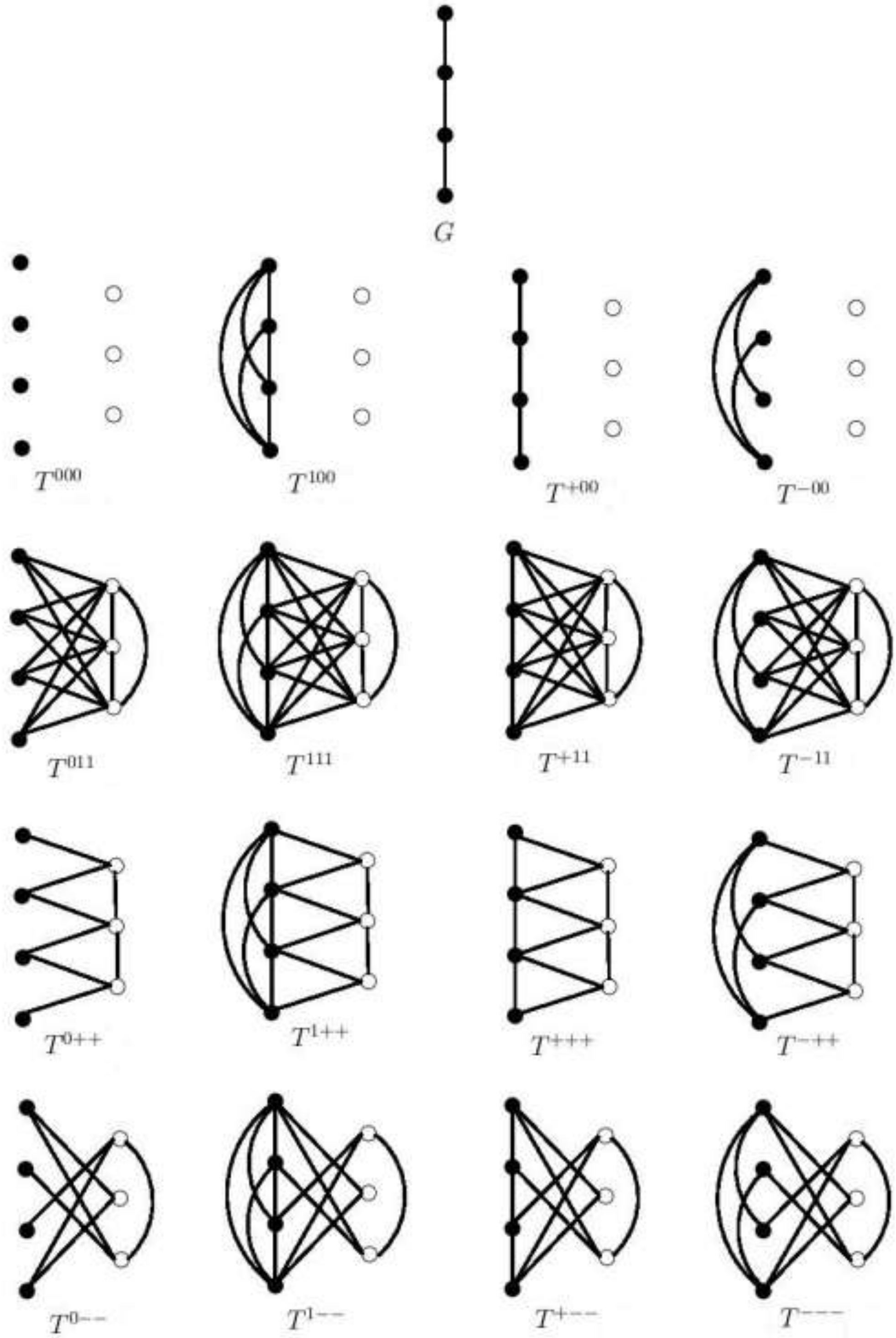
was encountered in [18], but remained unstudied for a long time by scholars doing research on degree$-$based topological indices. Furtula and Gutman, restudied this index recently and named it as *forgotten topological index*, or $F-$*index* [13].

Milic'evic' et al. [25] in 2004 reformulated the Zagreb indices in terms of edge-degrees instead of vertex-degrees. The *first and second reformulated Zagreb indices* are defined respectively as

$$EM_1 = EM_1(G) = \sum_{e\in E(G)} d_G(e)^2 = \sum_{e\sim f} [d_G(e) + d_G(f)] \text{ and}$$

$$EM_2 = EM_2(G) = \sum_{e\sim f} d_G(e)d_G(f).$$

Figure 1:



In [21], Hosamani and Trinajstić defined the *first and second reformulated Zagreb coindices*, respectively, as

$$\overline{EM_1} = \overline{EM_1}(G) = \sum_{e \nsim f} [d_G(e) + d_G(f)] \text{ and } \overline{EM_2} = \overline{EM_2}(G) = \sum_{e \nsim f} d_G(e) d_G(f).$$

The following theorems are useful for proving our main results.

Theorem 2.1 *[17] Let G be any graph of order n and size m. Then*

$$\overline{M_1}(G) = 2m(n-1) - M_1.$$

Theorem 2.2 *[17] Let G be a graph of order n and size m. Then*

$$\overline{M_2}(G) = 2m^2 - \left(\frac{1}{2}\right) M_1 - M_2.$$

Theorem 2.3 *[28] For any graph G of order n and size m,*

$$M_1(\overline{G}) = M_1 + n(n-1)^2 - 4m(n-1).$$

Theorem 2.4 *[17] Let G be a graph of order n and size m. Then*

$$M_2(\overline{G}) = \frac{1}{2} n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \left(\frac{2n-3}{2}\right) M_1 - M_2.$$

Theorem 2.5 *[1] Let G be a simple graph. Then $\overline{M_1}(G) = \overline{M_1}(\overline{G})$.*

Theorem 2.6 *[17] Let G be any graph of order n and size m. Then*

$$M_1(L(G)) = F - 4M_1 + 2M_2 + 4m.$$

Theorem 2.7 *[17] Let G be a graph of order n and size m. Then*

$$M_1(T_2(G)) = M_1(G^{++0}) = 4M_1 + 4m.$$

3 Vertex degree of T^{xyz}

The following theorems are obvious by the definition of T^{xyz} and are useful to find degree based topological indices.

Theorem 3.1 *Let G be a graph of order n, size m and let v be the point-vertex of T^{xyz} corresponding to a vertex v of G. Then*

$$d_{T^{xy0}}(v) = \begin{cases} 0 & \text{if } x = 0, y \in \{0,1,+,-\}. \\ n-1 & \text{if } x = 1, y \in \{0,1,+,-\}. \\ d_G(v) & \text{if } x = +, y \in \{0,1,+,-\}. \\ n-1-d_G(v) & \text{if } x = -, y \in \{0,1,+,-\}. \end{cases}$$

$$d_{T^{xy1}}(v) = \begin{cases} m & \text{if } x = 0, y \in \{0,1,+,-\}. \\ n+m-1 & \text{if } x = 1, y \in \{0,1,+,-\}. \\ m+d_G(v) & \text{if } x = +, y \in \{0,1,+,-\}. \\ n+m-1-d_G(v) & \text{if } x = -, y \in \{0,1,+,-\}. \end{cases}$$

$$d_{T^{xy+}}(v) = \begin{cases} d_G(v) & if \ x = 0, y \in \{0,1,+,-\}. \\ n-1+d_G(v) & if \ x = 1, y \in \{0,1,+,-\}. \\ 2d_G(v) & if \ x = +, y \in \{0,1,+,-\}. \\ n-1 & if \ x = -, y \in \{0,1,+,-\}. \end{cases}$$

$$d_{T^{xy-}}(v) = \begin{cases} m-d_G(v) & if \ x = 0, y \in \{0,1,+,-\}. \\ n+m-1-d_G(v) & if \ x = 1, y \in \{0,1,+,-\}. \\ m & if \ x = +, y \in \{0,1,+,-\}. \\ n+m-1-2d_G(v) & if \ x = -, y \in \{0,1,+,-\}. \end{cases}$$

Theorem 3.2 *Let* G *be a graph of order* n, *size* m *and let* e *be the line-vertex of* T^{xyz} *corresponding to an edge* e *of* G. *Then*

$$d_{T^{xy0}}(e) = \begin{cases} 0 & if \ y = 0, x \in \{0,1,+,-\}. \\ m-1 & if \ y = 1, x \in \{0,1,+,-\}. \\ d_G(e) & if \ y = +, x \in \{0,1,+,-\}. \\ m-1-d_G(e) & if \ y = -, x \in \{0,1,+,-\}. \end{cases}$$

$$d_{T^{xy1}}(e) = \begin{cases} n & if \ y = 0, x \in \{0,1,+,-\}. \\ n+m-1 & if \ y = 1, x \in \{0,1,+,-\}. \\ n+d_G(e) & if \ y = +, x \in \{0,1,+,-\}. \\ n+m-1-d_G(e) & if \ y = -, x \in \{0,1,+,-\}. \end{cases}$$

$$d_{T^{xy+}}(e) = \begin{cases} 2 & if \ y = 0, x \in \{0,1,+,-\}. \\ m+1 & if \ y = 1, x \in \{0,1,+,-\}. \\ 2+d_G(e) & if \ y = +, x \in \{0,1,+,-\}. \\ m+1-d_G(e) & if \ y = -, x \in \{0,1,+,-\}. \end{cases}$$

$$d_{T^{xy-}}(e) = \begin{cases} n-2 & if \ y = 0, x \in \{0,1,+,-\}. \\ n+m-3 & if \ y = 1, x \in \{0,1,+,-\}. \\ n-2+d_G(e) & if \ y = +, x \in \{0,1,+,-\}. \\ n+m-3-d_G(e) & if \ y = -, x \in \{0,1,+,-\}. \end{cases}$$

4 First Zagreb index of T^{xyz} graphs

Theorem 4.1 *Let* G *be a graph of order* n *and size* m. *Then*

1. $M_1(T^{000}) = 0$
2. $M_1(T^{100}) = n(n-1)^2$
3. $M_1(T^{+00}) = M_1$
4. $M_1(T^{-00}) = n(n-1)^2 - 4m(n-1) + M_1$
5. $M_1(T^{010}) = m(m-1)^2$

6. $M_1(T^{110}) = n(n-1)^2 + m(m-1)^2$
7. $M_1(T^{+10}) = m(m-1)^2 + M_1$
8. $M_1(T^{-10}) = n(n-1)^2 + m(m-1)^2 - 4m(n-1) + M_1$
9. $M_1(T^{0+0}) = F - 4M_1 + 2M_2 + 4m$
10. $M_1(T^{1+0}) = n(n-1)^2 + F - 4M_1 + 2M_2 + 4m$
11. $M_1(T^{++0}) = 4m - 3M_1 + 2M_2 + F$
12. $M_1(T^{-+0}) = n(n-1)^2 - 4m(n-2) - 3M_1 + 2M_2 + F$
13. $M_1(T^{001}) = nm(n+m)$
14. $M_1(T^{101}) = n(n-1)^2 + nm(3n+m-2)$
15. $M_1(T^{011}) = m[nm + (n+m-1)^2]$
16. $M_1(T^{111}) = (n+m)(n+m-1)^2$

Proof. Above expressions obtained by the definitions of first Zagreb index and T^{xyz} followed by Theorems 3.1, 3.2 and 1.1.

Theorem 4.2 *Let G be a graph of order n and size m. Then*
1. $M_1(T^{0-0}) = m[(m-1)(m+3)+4] - 2(m+1)M_1 + 2M_2 + F$
2. $M_1(T^{1-0}) = n(n-1)^2 + m[(m-1)(m+3)+4] - 2(m+1)M_1 + 2M_2 + F$
3. $M_1(T^{+-0}) = m(m-1)(m+3) + 4m - (2m+1)M_1 + 2M_2 + F$
4. $M_1(T^{--0}) = n(n-1)^2 + m(m-1)^2 + 4m(m-n+1) - (2m+1)M_1 + 2M_2 + F$.

Proof. 1. Note that $T^{0-0} \cong \overline{K_n} \cup \overline{L(G)}$.

Hence $M_1(T^{0-0}) = M_1(\overline{K_n}) + M_1(\overline{L(G)}) = M_1(\overline{L(G)})$.

From Theorem 2.3, we have $M_1(T^{0-0}) = M_1(L(G)) + (m-1)[m^2 + 3m - 2M_1]$.

Again from Theorem 2.6, we obtain the desired result.

Similarly, $T^{1-0} \cong K_n \cup \overline{L(G)}$, $T^{+-0} \cong G \cup \overline{L(G)}$ and $T^{--0} \cong \overline{G} \cup \overline{L(G)}$ we obtain the other three expressions.

Theorem 4.3 *Let G be a graph of order n and size m. Then*
1. $M_1(T^{+01}) = nm(n+m) + 4m^2 + M_1$
2. $M_1(T^{-01}) = (n+m-1)[n(n+m-1) - 4m] + n^2 m + M_1$
3. $M_1(T^{+11}) = m^2(n+4) + m(n+m-1)^2 + M_1$
4. $M_1(T^{-11}) = (n+m-1)[(n+m)(n+m-1) - 4m] + M_1$.

Proof. 1. $M_1(T^{+01}) = \sum_{w \in V(T^{+01})} d_{T^{+01}}(w)^2$

$$= \sum_{v \in V(T^{+01}) \cap V(G)} d_{T^{+01}}(v)^2 + \sum_{e \in V(T^{+01}) \cap E(G)} d_{T^{+01}}(e)^2$$

$$= \sum_{v \in V(G)} [m + d_G(v)]^2 + \sum_{e \in E(G)} n^2$$

$$= m^2 n + \sum_{v\in V(G)} d_G(v)^2 + 2m \sum_{v\in V(G)} d_G(v) + n^2 m.$$

2. $M_1(T^{-01}) \;=\; \sum_{v\in V(G)} [n+m-1-d_G(v)]^2 + \sum_{e\in E(G)} n^2$

$$= \sum_{v\in V(G)} \{(n+m-1)^2 + d_G(v)^2 - 2(n+m-1)d_G(v)\} + n^2 m$$

$$= n(n+m-1)^2 + \sum_{v\in V(G)} d_G(v)^2 - 2(n+m-1)\sum_{v\in V(G)} d_G(v) + n^2 m.$$

Similarly, one can obtain the expressions for $M_1(T^{+11})$ and $M_1(T^{-11})$.

Theorem 4.4 *Let* G *be a graph of order* n *and size* m. *Then*

1. $M_1(T^{-+1}) = (n+m-1)[n(n+m-1) - 4m] + nm(n-4) + 4m + (2n-3)M_1 + 2M_2 + F.$
2. $M_1(T^{+-1}) = m^2(n+4) + m[(n+m-1)(n+m+3)+4] - (2n+2m+1)M_1 + 2M_2 + F.$

Proof. 1. $M_1(T^{-+1}) \;=\; \sum_{v\in V(G)} [n+m-1-d_G(v)]^2 + \sum_{e\in E(G)} [n+d_G(e)]^2$

$$= (n+m-1)^2 n + \sum_{v\in V(G)} d_G(v)^2 - 2(n+m-1)\sum_{v\in V(G)} d_G(v) + \sum_{e\in E(G)} d_G(e)^2$$

$$+ n^2 m + 2n \sum_{e\in E(G)} d_G(e)$$

$$= (n+m-1)^2 n + M_1 - 4m(n+m-1) + M_1(L(G)) + n^2 m$$

$$+ 2n\left[2\left(-m + \frac{M_1}{2}\right)\right].$$

From Theorem 2.6, we obtain the result.

2. $M_1(T^{+-1}) \;=\; \sum_{v\in V(G)} [m+d_G(v)]^2 + \sum_{e\in E(G)} [n+m-1-d_G(e)]^2$

$$= \sum_{v\in V(G)} \{m^2 + d_G(v)^2 + 2m d_G(v)\}$$

$$+ \sum_{e\in E(G)} \{(n+m-1)^2 + d_G(e)^2 - 2(n+m-1)d_G(e)\}$$

$$= m^2 n + \sum_{v\in V(G)} d_G(v)^2 + 2m\sum_{v\in V(G)} d_G(v) + (n+m-1)^2 m + \sum_{e\in E(G)} d_G(e)^2$$

$$- 2(n+m-1)\sum_{e\in E(G)} d_G(e)$$

$$= m^2n + M_1 + 4m^2 + m(n+m-1)^2 + M_1(L(G)) - 2(n+m-1)\left[2\left(-m+\frac{1}{2}M_1\right)\right].$$

From Theorem 2.6, we obtain the result.

Theorem 4.5 *Let G be a graph of order n and size m. Then*

1. $M_1(T^{0+1}) = nm(n+m-4) + 4m + 2(n-2)M_1 + 2M_2 + F$
2. $M_1(T^{1+1}) = n[m(n-4) + (n+m-1)^2] + 4m + 2(n-2)M_1 + 2M_2 + F$
3. $M_1(T^{++1}) = nm(n+m-4) + 4m(m+1) + (2n-3)M_1 + 2M_2 + F$
4. $M_1(T^{0-1}) = m(nm+4) + m(n+m-1)(n+m+3) - 2(n+m+1)M_1 + 2M_2 + F$
5. $M_1(T^{1-1}) = (n+m)(n+m-1)^2 + 4m(n+m) - 2(n+m+1)M_1 + 2M_2 + F$
6. $M_1(T^{--1}) = (n+m)(n+m-1)^2 + 4m - (2n+2m+1)M_1 + 2M_2 + F$
7. $M_1(T^{10+}) = n(n-1)^2 + 4mn + M_1$
8. $M_1(T^{01+}) = m(m+1)^2 + M_1$
9. $M_1(T^{11+}) = (n-1)[n(n-1) + 4m] + m(m+1)^2 + M_1$
10. $M_1(T^{00-}) = m^2(n-4) + m(n-2)^2 + M_1$
11. $M_1(T^{01-}) = m^2(n-4) + m(n+m-3)^2 + M_1$
12. $M_1(T^{10-}) = n(n+m-1)^2 - 4m(n+m-1) + m(n-2)^2 + M_1$
13. $M_1(T^{11-}) = n(n+m-1)^2 - 4m(n+m-1) + m(n+m-3)^2 + M_1$.

Proof. Above expressions obtained by using definition of $M_1(T^{xyz})$ and Theorems 3.1, 3.2.

The expression for the first Zagreb index of some generalized xyz–Point-Line transformation graphs were obtained. We nevertheless state these below for the sake of completeness:

Theorem 4.6 *[17] Let G be any graph of order n and size m. Then*

1. $M_1(T^{00+}) = M_1(S(G)) = M_1 + 4m$
2. $M_1(T^{0++}) = M_1(T_1(G)) = M_1 + 2M_2 + F$
3. $M_1(T^{+0+}) = M_1(T_2(G)) = 4M_1 + 4m$.

Theorem 4.7 *[4] Let G be a graph of order n and size m. Then*

1. $M_1(T^{+0-}) = m^2n + m(n-2)^2$
2. $M_1(T^{-0+}) = n(n-1)^2 + 4m$
3. $M_1(T^{-0-}) = m(n-2)^2 + (n+m-1)[n(n+m-1) - 8m] + 4M_1$
4. $M_1(T^{+1-}) = m^3 + 3nm^2 + n^2m - 6m^2 - 6nm + 9m$
5. $M_1(T^{+1+}) = m(m+1)^2 + 4M_1$
6. $M_1(T^{-1+}) = n(n-1)^2 + m(m+1)^2$
7. $M_1(T^{-1-}) = (n+m-1)[(n+m)(n+m-1) - 12m] + 4[M_1 + m]$.

Theorem 4.8 *[6] Let G be a graph of order n and size m. Then*

1. $M_1(T^{0+-}) = m(n-4)(n+m-4) + (2n-7)M_1 + 2M_2 + F$

2. $M_1(T^{0-+}) = m(m+3)^2 - (2m+5)M_1 + 2M_2 + F$
3. $M_1(T^{0--}) = m^2(n-4) + m(n+m-1)^2 + (3-2n-2m)M_1 + 2M_2 + F$
4. $M_1(T^{1-+}) = m[mn-4m+(n-4)^2] + (n+m-1)[(n+m)(n+m-1) - 4m(n-3)]$
$- (2m+5)M_1 + 2M_2 + F$
5. $M_1(T^{1+-}) = m(m+3)^2 + (n+m-1)[n^2 - m^2 + 2mn - n - 11m]$
$+ (2n-7)M_1 + 2M_2 + F$
6. $M_1(T^{1++}) = m^2(n-4) + (n+2m)(n+m-1)^2 - 2m(n+m-1)(2n+m-3)$
$+ M_1 + 2M_2 + F$
7. $M_1(T^{1--}) = (n+m-1)[(n+m)(n+m-1) - 4m] - (2n+2m-3)M_1 + 2M_2 + F.$

The expressions for first Zagreb index of T^{+++}, T^{-++}, T^{+-+}, T^{--+}, T^{++-}, T^{-+-}, T^{+--}, T^{---} are obtained in [20]. However $M_1(T^{-+-})$ is incorrect.

Theorem 4.9 *[20] Let G be a simple graph of order n and size m. Then*

$$M_1(T^{-+-}) = (n+m)[n(n+m) - 2(n+4m)] + m[(n-4)^2 + 9] + 2(n-2)M_1 + 2M_2 + F.$$

The correct expression for $M_1(T^{-+-})$ is as follows;

Theorem 4.10 *Let G be a simple graph of order n and size m. Then*

$$M_1(T^{-+-}) = (n+m)[n(n+m) - 2(n+4m)] + m[(n-4)^2 + 8] + n + 2(n-2)M_1 + 2M_2 + F\ .$$

Note that from Theorem 2.1 and $M_1(T^{xyz})$, one can easily obtain the expressions for $\overline{M_1}(T^{xyz})$.

5 Second Zagreb index of T^{xyz} graphs

Theorem 5.1 *Let G be a graph of order n and size m. Then*
1. $M_2(T^{000}) = 0$
2. $M_2(T^{100}) = \frac{1}{2}n(n-1)^3$
3. $M_2(T^{+00}) = M_2$
4. $M_2(T^{-00}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \left(\frac{2n-3}{2}\right)M_1 - M_2$
5. $M_2(T^{010}) = \frac{1}{2}m(m-1)^3$
6. $M_2(T^{110}) = \frac{1}{2}\{n(n-1)^3 + m(m-1)^3\}$
7. $M_2(T^{+10}) = \frac{1}{2}m(m-1)^3 + M_2$

8. $M_2(T^{-10}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \left(\frac{2n-3}{2}\right)M_1 - M_2 + \frac{1}{2}m(m-1)^3$

9. $M_2(T^{0+0}) = EM_2$

10. $M_2(T^{1+0}) = \frac{1}{2}n(n-1)^3 + EM_2$

11. $M_2(T^{++0}) = M_2 + EM_2$

12. $M_2(T^{-+0}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \left(\frac{2n-3}{2}\right)M_1 - M_2 + EM_2$

13. $M_2(T^{001}) = n^2m^2$

14. $M_2(T^{101}) = \left(\frac{n(n+m-1)}{2}\right)[(n-1)(n+m-1) + 2nm]$

15. $M_2(T^{011}) = \left(\frac{m(n+m-1)}{2}\right)[2mn + (m-1)(n+m-1)]$

16. $M_2(T^{111}) = \left(\frac{(n+m-1)^2}{2}\right)[n(n-1) + 2mn + m(m-1)]$.

Proof. Above expressions obtained by the definition of second Zagreb index and T^{xyz} followed by Theorems 3.1, 3.2 and 1.1.

Theorem 5.2 *Let G be a graph of order n and size m. Then*

1. $M_2(T^{0-0}) = \frac{m(m-1)(m^2+4m+7)}{2} - \frac{3}{2}(m-1)(m+3)M_1 + \frac{1}{2}M_1^2 + \left(\frac{2m-3}{2}\right)(F + 2M_2) - EM_2$

2. $M_2(T^{1-0}) = \frac{n(n-1)^3}{2} + \frac{m(m-1)(m^2+4m+7)}{2} - \frac{3}{2}(m-1)(m+3)M_1 + \frac{1}{2}M_1^2$
$+\left(\frac{2m-3}{2}\right)(F + 2M_2) - EM_2$

3. $M_2(T^{+-0}) = \frac{m(m-1)(m^2+4m+7)}{2} - \frac{3}{2}(m-1)(m+3)M_1 + \frac{1}{2}M_1^2 + 2(m-1)M_2$
$+\left(\frac{2m-3}{2}\right)F - EM_2$

4. $M_2(T^{--0}) = (n-1)^2\left[\binom{n}{2} - 3m\right] + 2m^2 + \frac{m(m-1)(m^2+4m+7)}{2} + 2(m-2)M_2 + \left(\frac{2m-3}{2}\right)F$
$+\frac{M_1}{2}[2n - 3(m^2+2m-2)] + \frac{1}{2}M_1^2 - EM_2$

Proof. 1. $M_2(T^{0-0}) = M_2(\overline{K_n}) + M_2(\overline{L(G)}) = M_2(\overline{L(G)})$.
From Theorem 2.4 and followed by the Theorem 2.6, we obtain the desired result.
Similarly, other three expressions can be obtained.

Theorem 5.3 *Let G be a graph of order n and size m. Then*

1. $M_2(T^{+01}) = m^3 + m^2n(n+2) + mM_1 + M_2$

2. $M_2(T^{-01}) = m^2\left[\binom{n}{2}+2-m\right]+m(n-3)(n-1)^2-4m^2(n-1)+\frac{n(n-1)^3}{2}$

$+mn[n(n+m-1)-2m]+\left(\frac{2(n+m)-3}{2}\right)M_1-M_2.$

3. $M_2(T^{+11}) = m^3+\left(\frac{m(n+m-1)}{2}\right)[2m(n+2)+(m-1)(n+m-1)]+mM_1+M_2$

4. $M_2(T^{-11}) = m^2\left[\binom{n}{2}+2-m\right]+m(n-3)(n-1)^2-4m^2(n-1)+\frac{n(n-1)^3}{2}$

$+\frac{m(2n+m-1)(n+m-1)^2}{2}-2m^2(n+m-1)+\left(\frac{2(n+m)-3}{2}\right)M_1-M_2.$

Proof. 1. $M_2(T^{+01}) = \sum_{st\in E(T^{+01})} d_{T^{+01}}(s)\, d_{T^{+01}}(t)$

$$= \sum_{uv\in E(T^{+01})\cap E(G)} d_{T^{+01}}(u)d_{T^{+01}}(v)+\sum_{ue\in E(T^{+01})\cap E(K_{n,m})} d_{T^{+01}}(u)d_{T^{+01}}(e)$$

$$= \sum_{uv\in E(G)}[m+d_G(u)][m+d_G(v)]+\sum_{ue\in E(K_{n,m})}[m+d_G(u)]n$$

$$= \sum_{uv\in E(G)}\{m^2+m[d_G(u)+d_G(v)]+d_G(u)d_G(v)\}+m^2n^2+n\sum_{u\in V(G)} md_G(u)$$

$$= m^3+m\sum_{uv\in E(G)}[d_G(u)+d_G(v)]+\sum_{uv\in E(G)} d_G(u)d_G(v)+(mn)^2$$

$$+nm\sum_{u\in V(G)} d_G(u).$$

2. $M_2(T^{-01}) = \sum_{uv\in E(T^{-01})\cap E(\overline{G})}[m+d_{\overline{G}}(u)][m+d_{\overline{G}}(v)]+\sum_{uv\in E(T^{-01})\setminus E(\overline{G})}[n+m-1-d_G(u)]n$

$$= \sum_{uv\in E(\overline{G})}\left\{m^2+m[d_{\overline{G}}(u)+d_{\overline{G}}(v)]+d_{\overline{G}}(u)d_{\overline{G}}(v)\right\}$$

$$+\sum_{ue\in E(K_{n,m})}\{n(n+m-1)-nd_G(u)\}$$

$$= m^2\left[\binom{n}{2}-m\right]+m\sum_{uv\in E(\overline{G})}[d_{\overline{G}}(u)+d_{\overline{G}}(v)]+\sum_{uv\in E(\overline{G})} d_{\overline{G}}(u)d_{\overline{G}}(v)$$

$$+n(n+m-1)(mn)-n\sum_{u\in V(G)} md_G(u)$$

$$= m^2\left[\binom{n}{2}-m\right]+mM_1(\overline{G})+M_2(\overline{G})+n^2m(n+m-1)-2nm^2$$

From Theorems 2.3 and 2.4, we have

$$M_2(T^{-01}) = m^2\left[\binom{n}{2}+2-m\right]+m(n-3)(n-1)^2-4m^2(n-1)+\frac{n(n-1)^3}{2}$$
$$+\left(\frac{2(n+m)-3}{2}\right)M_1 - M_2 + n^2m(n+m-1)-nm(2m).$$

Similarly, we can obtain the expressions for $M_2(T^{+11})$ and $M_2(T^{-11})$.

Theorem 5.4 *Let* G *be a graph of order* n *and size* m. *Then*

1. $M_2(T^{0+1}) = n^2m(m-1)-2nm^2+nEM_1+EM_2+\left(\frac{n(n+2m)}{2}\right)M_1$
2. $M_2(T^{1+1}) = \binom{n}{2}(n+m-1)^2+mn(n-2)(n+m-1)-n^2m+nEM_1+EM_2$
$+\left(\frac{n[2(n+m-1)+n]}{2}\right)M_1$
3. $M_2(T^{++1}) = m^2(n^2+m-4)-n^2m+\frac{M_1}{2}[n^2+2m(n+3)]+M_2+nEM_1+EM_2$
4. $M_2(T^{-+1}) = m^2\left[\binom{n}{2}-2(3n-5)-m\right]+(n-1)^2\left[\binom{n}{2}+m(n-3)\right]+nEM_1+EM_2$
$+mn[(n-2)(n+m-1)-n]+\left(\frac{(n-1)[3(n+1)+2m]}{2}\right)M_1-M_2.$

Proof. 1. $M_2(T^{0+1}) = \sum_{e\sim e'}[n+d_G(e)][n+d_G(e')] + \sum_{ue\in E(K_{n,m})} m[n+d_G(e)]$

$$= \sum_{e\sim e'}\{n^2+n[d_G(e)+d_G(e')]+d_G(e)d_G(e')\}$$

$$+\sum_{ue\in E(K_{n,m})}[mn+md_G(e)]$$

$$= n^2\left[-m+\frac{1}{2}M_1\right]+n\sum_{e\sim e'}[d_G(e)+d_G(e')]+\sum_{e\sim e'}d_G(e)d_G(e')$$

$$+m^2n^2+m\sum_{e\in E(G)}nd_G(e)$$

$$= n^2\left(-m+\frac{1}{2}M_1\right)+nEM_1+EM_2+m^2n^2+mn\left[2\left(-m+\frac{1}{2}M_1\right)\right].$$

2. $M_2(T^{1+1}) = \sum_{uv\in E(K_n)}(n+m-1)^2+\sum_{ue\in E(K_{n,m})}(n+m-1)[n+d_G(e)]$

$$+\sum_{e\sim e'}[n+d_G(e)][n+d_G(e')]$$

$$=\binom{n}{2}(n+m-1)^2+n^2m(n+m-1)+(n+m-1)\sum_{e\in E(G)} nd_G(e)$$

$$+n^2\left[-m+\frac{M_1}{2}\right]+nEM_1+EM_2.$$

3. $$M_2(T^{++1})=\sum_{uv\in E(G)}[m+d_G(u)][m+d_G(v)]+\sum_{e\sim e'}[n+d_G(e)][n+d_G(e')]$$

$$+\sum_{ue\in E(K_{n,m})}[m+d_G(u)][n+d_G(e)]$$

$$=m^3+mM_1+M_2+n^2\left(-m+\frac{M_1}{2}\right)+nEM_1+EM_2+(mn)^2+nm\sum_{u\in V(G)} d_G(u)$$

$$+nm\sum_{e\in E(G)} d_G(e)+\sum_{ue\in E(K_{n,m})} d_G(u)d_G(e)$$

$$=m^3+mM_1+M_2+n^2\left(-m+\frac{M_1}{2}\right)+nEM_1+EM_2+m^2n^2$$

$$+2m^2n+mn\left[2\left(-m+\frac{M_1}{2}\right)\right]+2m(-2m+M_1).$$

4. $$M_2(T^{-+1})=\sum_{uv\in E(\overline{G})}[m+d_{\overline{G}}(u)][m+d_{\overline{G}}(v)]+\sum_{e\sim e'}[n+d_G(e)][n+d_G(e')]$$

$$+\sum_{ue\in E(K_{n,m})}[n+m-1-d_G(u)][n+d_G(e)]$$

$$=m^2\left[\binom{n}{2}-m\right]+mM_1(\overline{G})+M_2(\overline{G})+n^2\left(-m+\frac{M_1}{2}\right)+nEM_1+EM_2$$

$$+\sum_{ue\in E(K_{n,m})}\{n(n+m-1)+(n+m-1)d_G(e)-nd_G(u)-d_G(e)d_G(u)\}.$$

From Theorems 2.3 and 2.4, we have

$$M_2(T^{-+1})=m^2\left[\binom{n}{2}+2-m\right]+m(n-3)(n-1)^2-4m^2(n-1)+\frac{n(n-1)^3}{2}+\frac{[2(n+m)-3]}{2}M_1$$

$$-M_2+\left\{mn^2(n+m-1)+2n(n+m-1)\left(-m+\frac{M_1}{2}\right)-nm(2m)-2m(-2m+M_1)\right\}$$

$$+n^2\left(-m+\frac{1}{2}M_1\right)+nEM_1+EM_2\,.$$

Theorem 5.5 *Let* G *be a graph of order* n *and size* m. *Then*

1. $M_2(T^{0-1})=(n+m-1)^2\left[\binom{n+m}{2}-3\binom{n}{2}+3m\right]-\frac{M_1}{2}[(n+m)(3n+3m+2)-9]+2\left[\binom{n}{2}-m+\frac{M_1}{2}\right]^2$
$+\left(\frac{2(n+m)-3}{2}\right)[n(n-1)^2+F+2M_2+4m]-\left(\frac{n(n-1)^3}{2}\right)-EM_2.$

2. $M_2(T^{1-1})=(n+m-1)^2\left[\binom{n+m}{2}+3m\right]-2\left(\frac{M_1}{2}-m\right)^2+\left(\frac{2(n+m)-3}{2}\right)[F+2M_2+4m]$
$-\frac{M_1}{2}[3(n+m-1)^2+4(2n+2m-3)]-EM_2$

3. $M_2(T^{+-1})=(n+m-1)^2\left[\binom{n+m}{2}-3\binom{n}{2}+6m\right]+2\left[\binom{n}{2}-2m+\frac{M_1}{2}\right]^2-\frac{M_1}{2}[3(n+m)^2+2n-9]$
$+2(n+m-1)M_2+\left(\frac{2(n+m)-3}{2}\right)[n(n-1)^2-4m(n-2)+F]-2m^2-(n-1)^2\left[\binom{n}{2}-3m\right]-EM_2.$

4. $M_2(T^{--1})=(n+m-1)^2\binom{n+m}{2}+\frac{1}{2}M_1^2-\frac{3}{2}M_1[(n+m)^2-2]+\left(\frac{2(n+m)-3}{2}\right)(4m+F)$
$+2(n+m-2)M_2-EM_2.$

Proof. 1. As $T^{0-1}\cong\overline{T^{1+0}}$. From Theorems 2.4, 1.1 we have

$$M_2(T^{0-1})=M_2(\overline{T^{1+0}})\;=\frac{1}{2}(n+m)(n+m-1)^3-3\left[\binom{n}{2}-m+\frac{1}{2}M_1\right](n+m-1)^2$$

$$+2\left[\binom{n}{2}-m+\frac{1}{2}M_1\right]^2+\left(\frac{2(n+m)-3}{2}\right)M_1(T^{1+0})-M_2(T^{1+0})\,.$$

From Theorem 4.1 and Theorem 5.1, we have

$$M_2(T^{0-1})\;=\;\frac{1}{2}(n+m)(n+m-1)^3-3\left[\binom{n}{2}-m\right](n+m-1)^2-\frac{3}{2}M_1(n+m-1)^2$$

$$+2\left[\binom{n}{2}-m+\frac{M_1}{2}\right]^2+\left(\frac{2(n+m)-3}{2}\right)[n(n-1)^2+F-4M_1+2M_2+4m]$$

$$-\frac{n(n-1)^3}{2}-EM_2\,.$$

Similarly, we obtain the other expressions.

Theorem 5.6 *Let G be a graph of order n and size m. Then*

1. $M_2(T^{10+}) = \binom{n}{2}(n-1)^2 + 2m[(n-1)^2 + m] + 4m(n-1) + \frac{3}{2}M_1$
2. $M_2(T^{01+}) = (m+1)\left[M_1 + (m+1)\binom{m}{2}\right]$
3. $M_2(T^{11+}) = (n-1)^2\binom{n}{2} + (m+1)^2\binom{m}{2} + 2m(n-1)(n+m) + 2m^2 + \frac{(2m+1)}{2}M_1.$

Proof. 1. $M_2(T^{10+}) = \sum_{uv\in E(K_n)} [n-1+d_G(u)][n-1+d_G(v)] + \sum_{u\sim e} 2[n-1+d_G(u)]$

$$= \sum_{uv\in E(K_n)} \{(n-1)^2 + (n-1)[d_G(u)+d_G(v)] + d_G(u)d_G(v)\}$$

$$+ \sum_{u\sim e} 2(n-1) + 2\sum_{u\in V(G)} d_G(u)^2$$

$$= (n-1)^2\binom{n}{2} + (n-1)[M_1 + \overline{M_1}] + M_2 + \overline{M_2} + 4m(n-1) + 2M_1.$$

From Theorems 2.1 and 2.2, we have

$$M_2(T^{10+}) = \binom{n}{2}(n-1)^2 + 2m(n-1)^2 + 2m^2 - \frac{M_1}{2} + 4m(n-1) + 2M_1.$$

2. $M_2(T^{01+}) = \sum_{u\sim e} (m+1)d_G(u) + \sum_{ee'\in E(K_m)} (m+1)^2$

$$= (m+1)\sum_{u\in V(G)} d_G(u)^2 + (m+1)^2\binom{m}{2}$$

$$= (m+1)M_1 + (m+1)^2\binom{m}{2}.$$

3. $M_2(T^{11+}) = \sum_{uv\in E(K_n)} [n-1+d_G(u)][n-1+d_G(v)] + \sum_{ee'\in E(K_m)} (m+1)^2$

$$+ \sum_{u\sim e} (m+1)\,[n-1+d_G(u)]$$

$$= (n-1)^2\binom{n}{2} + (n-1)[M_1 + \overline{M_1}] + M_2 + \overline{M_2} + (m+1)^2\binom{m}{2}$$

$$+ (m+1)\{(n-1)(2m) + M_1\}.$$

From Theorems 2.1 and 2.2, we have

$$M_2(T^{11+}) = \left\{(n-1)^2\binom{n}{2} + 2m(n-1)^2 + 2m^2 - \frac{M_1}{2}\right\} + 2m(n-1)(m+1) + (m+1)M_1$$

$$+ (m+1)^2\binom{m}{2}.$$

Theorem 5.7 *Let G be a graph of order n and size m. Then*

1. $M_2(T^{00-}) = (n-2)[m^2(n-4) + M_1]$

2. $M_2(T^{10-}) = \binom{n}{2}(n+m-1)^2 - 2m^2(n-3) + m(n+m-1)[(n-2)^2 - 2(n-1)] + \left(\frac{2n-5}{2}\right)M_1$

3. $M_2(T^{01-}) = \binom{m}{2}(n+m-3)^2 + m^2(n-4)(n+m-3) + (n+m-3)M_1$

4. $M_2(T^{11-}) = \binom{n}{2}(n+m-1)^2 + \binom{m}{2}(n+m-3)^2 - 2m^2(n+m-4) + \frac{[2(n+m)-7]}{2}M_1$
$+ m(n+m-1)[(n-2)(n+m-3) - 2(n-1)].$

Proof. 1. $M_2(T^{00-}) = \sum_{u \nsim e}(n-2)[m - d_G(u)]$

$$= m^2(n-2)^2 - (n-2)\left(m\sum_{u\in V(G)} d_G(u) - \sum_{u\in V(G)} d_G(u)^2\right)$$

$$= m^2(n-2)^2 - (n-2)[2m^2 - M_1].$$

2. $$M_2\left(T^{10-}\right) = \sum_{uv\in E(K_n)}[n+m-1-d_G(u)][n+m-1-d_G(v)]$$

$$+ \sum_{u \nsim e}(n-2)[n+m-1-d_G(u)]$$

$$= (n+m-1)^2\binom{n}{2} + (n+m-1)[M_1 + \overline{M_1}] + M_2 + \overline{M_2}$$

$$+ m(n-2)^2(n+m-1) - (n-2)[2m^2 - M_1].$$

From Theorems 2.1 and 2.2, we have

$$M_2\left(T^{10-}\right)=\binom{n}{2}(n+m-1)^2-2m(n-1)(n+m-1)+2m^2-\frac{M_1}{2}$$

$$+m(n+m-1)(n-2)^2-2m^2(n-2)+(n-2)M_1.$$

The expressions for $M_2(T^{01-})$ and $M_2(T^{11-})$ can be obtained.

Theorem 5.8 *Let G be a graph of order n and size m. Then*

1. $M_2\left(T^{++-}\right)=m^3+m(n-2)(mn-n-4m+2)+(n-2)EM_1+EM_2$
$+\dfrac{[(n-2)(n+2m-2)]}{2}M_1$

2. $M_2\left(T^{+-+}\right)=(m+1)^2\binom{m+1}{2}-(m+1)\overline{EM_1}+\overline{EM_2}-2F-\dfrac{(m^2-2m-11)}{2}M_1$

3. $M_2\left(T^{-+-}\right)=(n+m-1)^2\left[\binom{n}{2}-m\right]+(n-2)^2\left[\dfrac{M_1}{2}-m\right]+m(n-2)(n-4)(n+m-1)$
$-4m^2(n-4)+M_1[(n-2)(n+m+1)-4(m+1)]+4M_2+2F$
$-2(n+m-1)\overline{M_1}+4\overline{M_2}+(n-2)EM_1+EM_2.$

Proof. 1. $M_2\left(T^{++-}\right)=\sum_{uv\in E(G)}(m^2)+\sum_{e\sim e'}[d_G(e)+n-2][d_G(e')+n-2]$

$$+\sum_{u\nsim e(=ab)} m[d_G(a)+d_G(b)+n-4]$$

$$=m^3+\sum_{e\sim e'}\{d_G(e)d_G(e')+(n-2)[d_G(e)+d_G(e')]+(n-2)^2\}$$

$$+m(n-2)\sum_{ab\in E(G)}[d_G(a)+d_G(b)]+m(n-4)[m(n-2)]$$

$$=m^3+\sum_{e\sim e'}d_G(e)d_G(e')+(n-2)\sum_{e\sim e'}[d_G(e)+d_G(e')]$$

$$+(n-2)^2\left(-m+\frac{M_1}{2}\right)+m(n-2)M_1+m^2(n-2)(n-4)$$

$$=m^3+EM_2+(n-2)EM_1+(n-2)^2\left(-m+\frac{M_1}{2}\right)+m(n-2)M_1$$

$+m^2(n-2)(n-4).$

$$2.\ M_2(T^{+-+}) = \sum_{uv\in E(G)} [2d_G(u)][2d_G(v)] + \sum_{e \nsim e'} [m+1-d_G(e)][m+1-d_G(e')]$$

$$+ \sum_{u\ \sim e(=uv)} 2d_G(u)[m+3-d_G(u)-\ d_G(v)]$$

$$=4\sum_{uv\in E(G)} d_G(u)d_G(v) + \sum_{e \nsim e'}\{(m+1)^2-(m+1)[d_G(e)+d_G(e')]+d_G(e)\,d_G(e')\}$$

$$+\sum_{uv\in E(G)} [2d_G(u)+2d_G(v)][m+3-d_G(u)-d_G(v)]$$

$$=4M_2+(m+1)^2\left[\binom{m+1}{2}-\frac{M_1}{2}\right]-(m+1)\overline{EM_1}+\overline{EM_2}$$

$$+2(m+3)\sum_{uv\in E(G)}[d_G(u)+d_G(v)]-2\sum_{uv\in E(G)}[d_G(u)+d_G(v)]^2$$

$$=4M_2+(m+1)^2\left[\binom{m+1}{2}-\frac{M_1}{2}\right]-(m+1)\overline{EM_1}+\overline{EM_2}+2(m+3)M_1$$

$$-2\left(\sum_{uv\in E(G)}[d_G(u)^2+d_G(v)^2]+2\sum_{uv\in E(G)} d_G(u)d_G(v)\right).$$

$$3.\ M_2(T^{-+-}) = \sum_{uv\in E(\overline{G})}[n+m-1-2d_G(u)][n+m-1-2d_G(v)]$$

$$+\sum_{e\sim e'}[n-2+d_G(e)][n-2+d_G(e')]$$

$$+\sum_{u\nsim e}[n+m-1-2d_G(u)][n-2+d_G(e)]$$

$$=\sum_{uv\in E(\overline{G})}\{(n+m-1)^2-2(n+m-1)[d_G(u)+d_G(v)]+4d_G(u)d_G(v)\}$$

$$+(n-2)^2\left(-m+\frac{M_1}{2}\right)+(n-2)EM_1+EM_2$$

$$+\sum_{u\nsim e}\{(n+m-1)(n-2)+(n+m-1)d_G(e)-2(n-2)d_G(u)-2d_G(u)d_G(e)\}$$

$$=(n+m-1)^2\left[\binom{n}{2}-m\right]-2(n+m-1)\sum_{u\nsim v}[d_G(u)+d_G(v)]$$

$$+ 4 \sum_{u \nsim v} d_G(u) d_G(v) + (n-2)^2\left(-m+\frac{M_1}{2}\right) + (n-2)EM_1$$

$$+ EM_2 + m(n+m-1)(n-2)^2 + (n-2)(n+m-1)\left[2\left(-m+\frac{M_1}{2}\right)\right]$$

$$-2(n-2)[2m^2 - M_1] - 2\left\{4m\left(-m+\frac{M_1}{2}\right) - F - 2M_2 + 2M_1\right\}.$$

Theorem 5.9 *Let G be a graph of order n and size m. Then*

1. $M_2\left(T^{-++}\right) = (n-1)^2\left[\binom{n}{2} - m\right] + EM_2 + 2EM_1 + 2nM_1 - 4m$

2. $M_2\left(T^{--+}\right) = (n-1)^2\left[\binom{n}{2} - m\right] + 2m(n-1)(m+3) + (m+1)^2\binom{m+1}{2}$
$-\dfrac{(m^2+2m+4n-3)}{2} M_1 - (m+1)\,\overline{EM_1} + \overline{EM_2}$

3. $M_2\left(T^{+--}\right) = m^3 + \dfrac{(n+m-3)^2}{2}[m(m+1) - M_1] + m(n-2)[m(n+m-1) - M_1]$
$-(n+m-3)\overline{EM_1} + \overline{EM_2}$

4. $M_2\left(T^{---}\right) = (n+m-1)^2\left[\binom{n+m}{2} - 3m_T\right] + 2\left[m_T^2 + 2m\right] + 4(n+m-2)M_1$
$+(2n+2m-11)M_2 + \left(\dfrac{2n+2m-7}{2}\right)F - 2EM_1 - EM_2,$

where m_T is the number of edges of T^{+++}.

Proof. Above expressions obtained by using definition of $M_2(T^{xyz})$ and Theorems 3.1, 3.2.

The expression for the second Zagreb index of some generalized xyz–Point-Line transformation graphs were obtained. We nevertheless state these below for the sake of completeness:

Theorem 5.10 *[5] Let G be a graph of order n and size m. Then*
1. $M_2\left(T^{00+}\right) = 2M_1$
2. $M_2\left(T^{0++}\right) = 2EM_1 + EM_2 + 2[M_1 + M_2] + F - 4m$
3. $M_2\left(T^{+++}\right) = 2M_1 + 8M_2 + 2EM_1 + EM_2 + 2F - 4m.$

Theorem 5.11 *[4] Let G be a graph of order n and size m. Then*
1. $M_2\left(T^{+0+}\right) = 4[M_1 + M_2]$
2. $M_2\left(T^{+0-}\right) = m^3 + m^2(n-2)^2$

3. $M_2\left(T^{-0+}\right)=\dfrac{(n-1)}{2}[n(n-1)^2-2m(n-1)+8m]$

4. $M_2\left(T^{-0-}\right)=(n+m-1)\left\{(n+m-1)\left[\binom{n}{2}-m\right]+m(n-2)^2-2\overline{M_1}\right\}+4\overline{M_2}$

$-2(n-2)(2m^2-M_1)$

5. $M_2(T^{+1-})=\dfrac{1}{2}[4nm^3+3n^2m^2-18nm^2-n^2m+6nm-9m^3+m^4+27m^2-9m]$

6. $M_2\left(T^{+1+}\right)=2(n+m-1)\,\overline{M_1}-4\overline{M_2}+2M_1(n+2m-1)$

$+\dfrac{m}{2}[23m-8nm-8n^2+16n-9+m^2+m^3]$

7. $M_2\left(T^{-1+}\right)=\dfrac{1}{2}[n^4+m^4-3n^3+m^3+4nm^2-2n^2m+8nm+3n^2-5m^2-7m-n]$

8. $M_2\left(T^{-1-}\right)=2m[11m+2n-3]+2(2n+2m-5)M_1-4M_2$

$+(n+m-1)^2\left[\binom{n+m}{2}-9m\right].$

Theorem 5.12 *[6] Let G be a graph of order n and size m. Then*

1. $M_2\left(T^{1++}\right)=(n-1)^2\left[\binom{n}{2}+2m\right]+2m(m-2)+\dfrac{M_1}{2}(4n-1)+F+2M_2+2EM_1+EM_2$

2. $M_2\left(T^{0-+}\right)=(m+1)^2\binom{m+1}{2}-\dfrac{(m^2-5)}{2}M_1-(m+1)\overline{EM_1}+\overline{EM_2}-F-2M_2$

3. $M_2\left(T^{1-+}\right)=(n-1)^2\binom{n}{2}+(m+1)^2\binom{m+1}{2}+2m[(n-1)(n+m+2)+m]$

$-\dfrac{M_1}{2}[m^2+4(n-2)]-F-2M_2-(m+1)\overline{EM_1}+\overline{EM_2}$

4. $M_2\left(T^{0+-}\right)=m^2[(n-2)(n-6)+4]-m(n-2)^2+\dfrac{M_1}{2}[2(m+1)(n-4)+(n-2)^2]$

$+F+2M_2+(n-2)EM_1+EM_2$

5. $M_2\left(T^{1+-}\right)=(n+m-1)^2\binom{n}{2}+m(n+m-1)[(n-2)(n-4)-2(n-1)]+2M_2+F$

$-2m^2(n-5)-m(n-2)^2+\dfrac{M_1}{2}[(n-2)^2-1]+M_1[(n-2)(n+m)-2(m+1)]+(n-2)EM_1+EM_2$

6. $M_2\left(T^{0--}\right)=(n+m-3)^2\left[\binom{m+1}{2}-\dfrac{M_1}{2}\right]+m^2(n-4)(n+m-1)$

$-[m(n-5)-n+1]M_1-F-2M_2-(n+m-3)\,\overline{EM_1}+\overline{EM_2}$

7. $M_2\left(T^{1--}\right)=(n+m-1)^2\binom{n}{2}+(n+m-3)^2\binom{m+1}{2}-2m^2(n+m-2)-2M_2-F$

$+m(n+m-1)[(n-2)(n+m-1)-2(n-1)]-\dfrac{M_1}{2}[(n+m-3)^2+1]$

$-M_1[(n-2)(n+m-1)-(n+3m-1)]-(n+m-3)\overline{EM_1}+\overline{EM_2}.$

Note that from Theorem 2.2 and using expressions of $M_1(T^{xyz})$ and $M_2(T^{xyz})$, we can easily obtain the expressions for $\overline{M_2}(T^{xyz})$.

Acknowledgement[*] This research is supported by UGC-SAP DRS-III, New Delhi, India for 2016-2021: F.510/3/DRS-III/2016(SAP-I) Dated: 29^{th} Feb. 2016.

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