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CHARACTERIZATION OF TENSOR NORMS AND CONVERGENCE IN C^*-ALGEBRAS

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Abstract

A linear map ϕ from a C^*-algebra $\mathcal{A}$ to a C^*-algebra $\mathcal{B}$ is positive if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. ϕ is completely positive if for the corresponding linear maps ϕ_n from the C^*-algebra of n by n matrices with entries from $\mathcal{A}$ to the C^*-algebra of n by n matrices with entries from $\mathcal{B}$, ϕ_n is positive for all natural numbers n. ϕ_n is completely bounded if every ϕ_n is bounded and the supremum of the norm of ϕ_n is finite for all natural numbers n. In this paper we have considered the C^*-algebras of n by n matrices, constructed various maps between the C^*-algebras and characterized the cross-norms of the C^*-algebras. We have established the conditions for complete positivity and complete boundedness of the tensor product of the maps on the C^*-algebras.

Keywords : C^*-algebras, Tensor products and Tensor cross-norms.

0.1 Introduction

The development of the theory of C^*-algebras is an area that has attracted a lot of concern from many Mathematicians. In 1955, Stinespring obtained a theorem characterizing certain operator valued positive maps on C^*-algebra in terms of representations of those C^*-algebras, what is called Stinespring Representation theorem [9], [17] and asserted that if $\mathcal{A}$ is a unital C^*-algebra and $\phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a completely positive map, then there exists a Hilbert space $\mathcal{K}$, a bounded operator $V : \mathcal{H} \to \mathcal{K}$ and a unital $*$-homomorphism, $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\phi(a) = V^*\pi(a)V$, for every $a \in \mathcal{A}$ and that $\|\phi\|_{cb} = \|\phi(1)\| = \|V^*V\| = \|V\|^2$, [17].

Naimark showed that every C^*-algebra can be faithfully represented as a subalgebra of $B(\mathcal{H})$. Every representation π of $\mathcal{A}$ on $B(\mathcal{H})$ and vector $x \in \mathcal{H}$

defines a linear functional f on $\mathcal{A}$ by $f(a) = \langle \pi(a)x, x\rangle$. Such a functional is positive, (and is automatically continuous and contractive). There exists a Hilbert space $\mathcal{H}_f$, a vector $x_f \in \mathcal{H}_f$ and a representation π_f of $\mathcal{A}$ on $\mathcal{H}_f$ such that $f(a) = \langle \pi_f(a)x_f, x_f\rangle$,$\forall$ $a \in \mathcal{A}$, [14]. This construction is known as Gelfand-Naimark-Segal construction.

Let $\mathcal{H}$ be a Hilbert space, $B(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$ and $\mathcal{H}^{(n)}$ be the direct sum of $n-$copies of $\mathcal{H}$. If $M_n(B(\mathcal{H}))$, is the set of $n \times n$ matrices with entries from $B(\mathcal{H})$ and $B\left(\mathcal{H}^{(n)}\right)$ is the space of all bounded linear operators on $\mathcal{H}^{(n)}$, then there exist linear maps $\phi : M_n(B(\mathcal{H})) \to B\left(\mathcal{H}^{(n)}\right)$ such that ϕ is a $*-$isomorphism, for all $n \in \mathbb{N}$. Moreover, this ϕ is a representation of $M_n(B(\mathcal{H}))$ on the Hilbert space $\mathcal{H}^{(n)}$. Therefore, we can identify $M_n(B(\mathcal{H}))$ with $B\left(\mathcal{H}^{(n)}\right)$. Thus $M_n(B(\mathcal{H})) \cong B\left(\mathcal{H}^{(n)}\right)$. This identification gives a unique norm that makes the $*$-algebra $M_n(B(\mathcal{H}))$ a C^*-algebra [14]. This means that, for any subspace $\mathcal{A}$ of $B(H)$, $M_n(\mathcal{A})$ is considered as a subspace of $M_n(B(H)) \cong B(H^{(n)}$ and hence $M_n(\mathcal{A})$ is a C^*-algebra. Thus, $M_n(\mathcal{A}) \subseteq M_n(B(H))$ and hence $M_n(\mathcal{A})$ can be considered as an operator space.

Given a linear map $\phi : \mathcal{A} \to \mathcal{B}$, then we can define corresponding maps $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by
$\phi_n([a_{i,j}]) = [\phi(a_{i,j})]$ for all $n \in \mathbb{N}$, $[a_{i,j}] \in M_n(\mathcal{A})$. ϕ is positive if it maps positive elements of $\mathcal{A}$ to positive elements of $\mathcal{B}$. That is, if $\phi(\mathcal{A}^+) \subseteq \phi(\mathcal{B}^+)$. ϕ is completely positive if ϕ_n maps positive elements in $M_n(\mathcal{A})$ to positive elements in $M_n(\mathcal{B})$ for all $n \in \mathbb{N}$. ϕ is bounded if there is an $M \in \mathbb{R}$ such that $\|\phi([a_{i,j}])\| \leq M\|[a_{i,j}]\|$ and ϕ is completely bounded if each $\|\phi_n\|$ is bounded for every $n \in \mathbb{N}$ and that the completely bounded norm $\|\phi\|_{cb}$ is finite, that is, $\sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty$ and we set

$$\|\phi\|_{cb} = \sup\{\|\phi_n\| : n \in \mathbb{N}\} < \infty.$$

Let $T : X \to X$ be a map from the vector space X into itself, then the norm of T is given by $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1, x \in X\}$, [17].

Researchers have developed some forms of a completely positive map $\phi : M_n(\mathcal{A}) \longrightarrow M_n(\mathcal{B})$ on matrices. (Kraus) $\phi(A) = \sum_j V_j^* A V_j$, where V_j are matrices of the same appropriate size and (Choi) $\phi(A) = \sum_{i,j} \phi_{i,j} E_i^* A E_j$ where $\phi = [\phi_{i,j}]$ is a positive matrix and E_i are matrix units of appropriate sizes, [7], [5], [6].

If the vectors a_i and b_j are orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$, then the vectors $a_i \otimes b_j$ form an orthonormal basis of $\mathcal{H} \otimes \mathcal{K}$. The algebraic tensor product of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ has a natural positive definite sesquilinear form (scalar product) induced by the sesquilinear forms of $\mathcal{H}$ and $\mathcal{K}$. So in particular it has a natural positive definite quadratic form and the corresponding completion is a Hilbert space $\mathcal{H}\otimes\mathcal{K}$, called the (Hilbert space) tensor product of $\mathcal{H}$ and $\mathcal{K}$ [17]. If a norm $\|.\|_\gamma$ on $\mathcal{H}\otimes\mathcal{K}$ satisfies the condition

$$\|x_1 \otimes x_2\|_\gamma = \|x_1\|_\gamma \|x_2\|_\gamma \;\; \forall \;\; x_1 \in \mathcal{H}, x_2 \in \mathcal{K},$$

then it is called a cross-norm on $\mathcal{H}\otimes\mathcal{K}$. The completion of $\mathcal{H}\otimes\mathcal{K}$ under a cross-norm γ is denoted by $\mathcal{H}\otimes_\gamma\mathcal{K}$.

Takesaki [15] studied cross-norms and asserted that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital C^*-algebras and γ is a C^*-cross-norm on $\mathcal{A}_1\otimes\mathcal{A}_2$, then $\|x\|_{min} \leq \|x\|_\gamma$ for all $x \in \mathcal{A}_1\otimes\mathcal{A}_2$ and $\|x\|_{\max} \geq \|x\|_\gamma$

In our work, we have determine the tensor products of elements of the C^*-algebra of n by n matrices, determined and characterized the cross norms of these C^*-algebras in relation to the maps constructed between them. We have also investigate the conditions for complete positivity and complete boundedness of the tensor products of the constructed maps. We therefore arrange our work in the various sections in the following order:
1. Introduction; 2. Preliminaries; 3. Main Results.

0.2 Preliminaries

Definition 0.2.1. A **Banach $*$-algebra** is a $*$ -algebra $\mathcal{A}$ together with a complete submultiplicative norm such that

$$\|a^*\| = \|a\| \;\; (a \in A).$$

Definition 0.2.2. A C^*-algebra is a Banach $*$-algebra such that

$$\|a^*a\| = \|a^*\|^2 \quad (a \in \mathcal{A}).$$

Definition 0.2.3. An n-tuple of operators $(T1, ..., Tn)$ is said to doubly commute provided that $T_iT_j = T_jT_i$ and $T_iT_j^* = T_j^*T_i$ for all $i \neq j$.

Remark 0.2.4. Doubly commuting operators is a natural setting for generalizing the theory of spectral sets from a single-variable theory to a multivariable theory is the theory of tensor products of operator systems.

0.3 Main Results

Given an arbitrary C^*-algebra $\mathcal{A}$, by Gelfand Naimark Segal, $\mathcal{A}$ is a closed self-adjoint subalgebra of $B(H)$ for some Hilbert space H. This implies that $M_n(\mathcal{A})$ is a closed self-adjoint subalgebra of $M_n(B(H))$ and hence $M_n(\mathcal{A})$ is a C^*-algebra.

Let $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ be unital C^*-algebras of $n \times n$ matrices with entries from $\mathcal{A}$ and $\mathcal{B}$ respectively. Then their tensor product can be made into a $*$-algebra by setting $([T_{ij}] \otimes [S_{ij}])^* = [T_{ij}]^* \otimes [S_{ij}]^*$, where $[T_{ij}], [S_{ij}] \in M_n(\mathcal{A})$.

Let $[T_{ij}] \in M_n(B(H))$. Then the norm of $[T_{ij}]$ can be approximated by

$$\|[T_{ij}]\| \leq \sqrt{\sum_{i,j=1}^{n} \|T_{ij}\|^2}.$$

Therefore, if we take $\|.\|_2$ then we have,

$$\|[T_{ij}]\|_2 = \sqrt{\sum_{i,j=1}^{n} \|T_{ij}\|^2}.$$

Proposition 0.3.1. *Let $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$ be unital C^*-algebras of $n \times n$ matrices with entries from $\mathcal{A}$ and $\mathcal{B}$ respectively $[T_{ij}] \in M_n(\mathcal{A})$ and $[S_{ij}] \in M_n(\mathcal{B})$ be positive with nonnegative entries then, $[T_{ij}] \otimes [S_{ij}] \in M_n(\mathcal{A}) \otimes M_n(\mathcal{B})$, where $[T_{ij}] \otimes [S_{ij}]$ is a block matrix of size $n^2 \times n^2$ and $\|[T_{ij}] \otimes [S_{ij}]\| = \|[T_{ij}]\|\|[S_{ij}]\|$.*

Proof. Let $[T_{ij}] \in M_n(\mathcal{A})$ and $[S_{ij}] \in M_n(\mathcal{B})$ then,

$$[T_{ij}] \otimes [S_{ij}] = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & & T_{2n} \\ \vdots & & & \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \otimes \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & & S_{2n} \\ \vdots & & & \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} T_{11}S_{11} & T_{12}S_{11} & \cdots & T_{1n}S_{11} \\ T_{21}S_{11} & T_{22}S_{11} & & T_{2n}S_{11} \\ \vdots & & & \\ T_{n1}S_{11} & T_{n2}S_{11} & \cdots & T_{nn}S_{11} \end{pmatrix} & \cdots & \begin{pmatrix} T_{11}S_{11} & T_{12}S_{11} & \cdots & T_{1n}S_{11} \\ T_{21}S_{11} & T_{22}S_{11} & & T_{2n}S_{11} \\ \vdots & & & \\ T_{n1}S_{11} & T_{n2}S_{11} & \cdots & T_{nn}S_{11} \end{pmatrix} \\ \vdots & \cdots & \vdots \\ \begin{pmatrix} T_{11}S_{n1} & T_{12}S_{n1} & \cdots & T_{n1}S_{n1} \\ T_{21}S_{n1} & T_{22}S_{n1} & & T_{2n}S_{n1} \\ \vdots & & & \\ T_{n1}S_{n1} & T_{n2}S_{n1} & \cdots & T_{nn}S_{n1} \end{pmatrix} & \cdots & \begin{pmatrix} T_{11}S_{nn} & T_{12}S_{nn} & \cdots & T_{1n}S_{nn} \\ T_{21}S_{nn} & T_{22}S_{nn} & & T_{2n}S_{nn} \\ \vdots & & & \\ T_{n1}S_{nn} & T_{n2}S_{nn} & \cdots & T_{nn}S_{nn} \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} T_{11}S_{11} & T_{12}S_{11} & \cdots & T_{1n}S_{11} & \cdots & T_{11}S_{1n} & T_{12}S_{1n} & \cdots & T_{1n}S_{1n} \\ T_{21}S_{11} & T_{22}S_{11} & & T_{2n}S_{11} & & T_{2n}S_{1n} & T_{22}S_{1n} & & T_{2n}S_{1n} \\ \vdots & & & & & \vdots & & & \\ T_{n1}S_{11} & T_{n2}S_{11} & \cdots & T_{nn}S_{11} & & T_{n1}S_{1n} & T_{n2}S_{1n} & \cdots & T_{nn}S_{1n} \\ \vdots & & & & \cdots & & & \vdots & \\ T_{11}S_{n1} & T_{12}S_{n1} & \cdots & T_{n1}S_{n1} & & T_{11}S_{nn} & T_{12}S_{nn} & \cdots & T_{1n}S_{nn} \\ T_{21}S_{n1} & T_{22}S_{n1} & & T_{2n}S_{n1} & & T_{21}S_{nn} & T_{22}S_{nn} & & T_{2n}S_{nn} \\ \vdots & & & & & \vdots & & & \\ T_{n1}S_{n1} & T_{n2}S_{n1} & \cdots & T_{nn}S_{n1} & & T_{n1}S_{nn} & T_{n2}S_{nn} & \cdots & T_{nn}S_{nn} \end{pmatrix}.$$

Since T_{ij} and S_{ij} are nonnegative, we have $\|[T_{ij}]\otimes[S_{ij}]\| = \left(\sum_{i,j=1}^{n} \|T_{ij}S_{ij}\|^2\right)^{\frac{1}{2}} = \left(\sum_{i,j=1}^{n}(\|T_{ij}\|\|S_{ij}\|)^2\right)^{\frac{1}{2}} = \left(\sum_{i,j=1}^{n}(\|T_{ij})^2\right)^{\frac{1}{2}} \left(\sum_{i,j=1}^{n}(\|S_{ij}\|)^2\right)^{\frac{1}{2}} = \|[T_{ij}]\|\|[S_{ij}]\|$. $\square$

Remark 0.3.2. Suppose $\mathcal{H}^{(n)}$ and $\mathcal{K}^{(n)}$ are the direct sums of n-copies of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, $h, h' \in \mathcal{H}^{(n)}$ and $k, k' \in \mathcal{K}^{(n)}$, then we obtain an inner product on $\mathcal{H}^{(n)} \otimes \mathcal{K}^{(n)}$ by setting

$$\langle h \otimes k, h' \otimes k' \rangle = \langle h, h' \rangle_{\mathcal{H}^{(n)}} \langle k, k' \rangle_{\mathcal{K}^{(n)}}.$$

Thus the completion of $\mathcal{H}^{(n)} \otimes \mathcal{K}^{(n)}$ with respect to this inner product is a Hilbert space.

Proposition 0.3.3. *Let $\mathcal{H}^{(n)}$ and $\mathcal{K}^{(n)}$ be the direct sums of n-copies of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, $h \in \mathcal{H}^{(n)}$ and $k \in \mathcal{K}^{(n)}$. Let also*

$[T_{ij}] \in M_n(B(H))$ and $[S_{ij}] \in M_n(B(K))$ be operators on $\mathcal{H}^{(n)}$ and $\mathcal{K}^{(n)}$ such that $[T_{ij}]h = h$, that is $M_n(B(H)) \mapsto B(\mathcal{H}^{(n)})$. Then, $([T_{ij}] \otimes [S_{ij}])(h \otimes k) = ([T_{ij}]h) \otimes ([S_{ij}]k)$ defines a bounded linear operator on $\mathcal{H}^{(n)} \otimes \mathcal{K}^{(n)}$.

Proof. Let $\mathcal{H}^{(n)}$ and $\mathcal{K}^{(n)}$ be the direct sums of n-copies of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ respectively, $h \in \mathcal{H}^{(n)}$ and $k \in \mathcal{K}^{(n)}$. Then for any arbitrary operators $[T_{ij}] \in M_n(B(\mathcal{H}))$ and $[S_{ij}] \in M_n(B(\mathcal{K}))$, we have

$$
\begin{aligned}
([T_{ij}]h) \otimes ([S_{ij}]k) &= \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & & & \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} \otimes \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & & & \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} \\
&= \begin{pmatrix} T_{11}h_1 + T_{12}h_2 + \cdots + T_{1n}h_n \\ T_{21}h_1 + T_{22}h_2 + \cdots + T_{2n}h_3 \\ \vdots \\ T_{n1}h_1 + T_{n2}h_2 + \cdots + T_{nn}h_n \end{pmatrix} \otimes \begin{pmatrix} S_{11}k_1 + S_{12}k_2 + \cdots + S_{1n}k_n \\ S_{21}k_1 + S_{22}k_2 + \cdots + S_{2n}k_3 \\ \vdots \\ S_{n1}k_1 + S_{n2}k_2 + \cdots + S_{nn}k_n \end{pmatrix} \\
&= \begin{pmatrix} \sum_{j=1}^n T_{1j}h_j \\ \sum_{j=1}^n T_{2j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix} \otimes \begin{pmatrix} \sum_{j=1}^n S_{1j}k_j \\ \sum_{j=1}^n S_{2j}k_j \\ \vdots \\ \sum_{j=1}^n S_{nj}k_j \end{pmatrix} \\
&= \begin{pmatrix} \begin{pmatrix} \sum_{j=1}^n T_{1j}k_j \\ \sum_{j=1}^n T_{2j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix} \sum_{j=1}^n S_{1j}k_j \\ \begin{pmatrix} \sum_{j=1}^n T_{1j}h_j \\ \sum_{j=1}^n T_{2j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix} \sum_{j=1}^n S_{2j}k_j \\ \vdots \\ \begin{pmatrix} \sum_{j=1}^n T_{1j}h_j \\ \sum_{j=1}^n T_{2j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{pmatrix} \sum_{j=1}^n S_{nj}k_j \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} \sum_{j=1}^{n} T_{1j}h_j \sum_{j=1}^{n} S_{1j}k_j \\ \sum_{j=1}^{n} T_{2j}h_j \sum_{j=1}^{n} S_{1j}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}h_j \sum_{j=1}^{n} S_{1j}k_j \\ \sum_{j=1}^{n} T_{1j}h_j \sum_{j=1}^{n} S_{2j}k_j \\ \sum_{j=1}^{n} T_{2j}h_j \sum_{j=1}^{n} S_{2j}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}h_j \sum_{j=1}^{n} S_{2j}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{1j}h_j \sum_{j=1}^{n} S_{nj}k_j \\ \sum_{j=1}^{n} T_{2j}h_j \sum_{j=1}^{n} S_{nj}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}h_j \sum_{j=1}^{n} S_{nj}k_j \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} T_{1j}h_jS_{1j}k_j \\ \sum_{j=1}^{n} T_{2j}h_jS_{1j}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}h_jS_{1j}k_j \\ \sum_{j=1}^{n} T_{1j}h_jS_{2j}k_j \\ \sum_{j=1}^{n} T_{2j}h_jS_{2j}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}h_jS_{2j}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{1j}h_jS_{nj}k_j \\ \sum_{j=1}^{n} T_{2j}h_jS_{nj}k_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}h_jS_{nj}k_j \end{pmatrix}$$

$\|([T_{ij}]h)\otimes([S_{ij}]k)\| = \left(\sum_{i,j,l=1}^{n} \left\|\sum_{i,j,l=1}^{n} T_{ij}h_jS_{lj}k_j\right\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i,j,l=1}^{n}\sum_{i,j,l=1}^{n} \|T_{ij}h_jS_{lj}k_j\|^2\right)^{\frac{1}{2}}$
$= \left(\sum_{i,j,l=1}^{n} \|T_{ij}h_jS_{lj}k_j\|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|\|h_j\|\|k_j\|)^2\right)^{\frac{1}{2}} =$
$\left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|)^2(\|h_j\|\|k_j\|)^2\right)^{\frac{1}{2}} = \left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|)^2\sum_{j=1}^{n}(\|h_j\|\|k_j\|)^2\right)^{\frac{1}{2}} =$
$\left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}(\|h_j\|\|k_j\|)^2\right)^{\frac{1}{2}} = \left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|)^2\right)^{\frac{1}{2}} \|h\otimes k\|$.

If we let $c = \left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|)^2\right)^{\frac{1}{2}} < \infty$, then we obtain $\|([T_{ij}]\otimes[S_{ij}])(h\otimes k)\| = \|([T_{ij}]h)\otimes([S_{ij}]k)\| \leq \left(\sum_{i,j,l=1}^{n}(\|T_{ij}\|\|S_{lj}\|)^2\right)^{\frac{1}{2}}\|h\otimes k\| = c\|h\otimes k\|$. Thus $\|([T_{ij}]\otimes[S_{ij}])(h\otimes k)\| \leq c\|h\otimes k\|$, hence $[T_{ij}]\otimes[S_{ij}]$ is bounded.

We now show linearity. Let α, β be scalars and $h, k \in h\otimes k$. Then h and

k are n^2-dimensional elements of $B(\mathcal{H}^{(n)})$ and $B(\mathcal{K}^{(n)})$ respectively.

$$\begin{aligned}
([T_{ij}]\otimes[S_{ij}])\left(\alpha\begin{pmatrix} h_1\\ h_2\\ \vdots\\ h_{n^2}\end{pmatrix}+\beta\begin{pmatrix} k_1\\ k_2\\ \vdots\\ k_{n^2}\end{pmatrix}\right) &= ([T_{ij}]\otimes[S_{ij}])\left(\begin{pmatrix} \alpha h_1\\ \alpha h_2\\ \vdots\\ \alpha h_{n^2}\end{pmatrix}+\begin{pmatrix} \beta k_1\\ \beta k_2\\ \vdots\\ \beta k_{n^2}\end{pmatrix}\right)\\
&= ([T_{ij}]\otimes[S_{ij}])\begin{pmatrix} \alpha h_1+\beta k_1\\ \alpha h_2+\beta k_2\\ \vdots\\ \alpha h_{n^2}+\beta k_{n^2}\end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix}
T_{11}S_{11} & T_{12}S_{11} & \cdots & T_{1n}S_{11} & \cdots & T_{11}S_{1n} & T_{12}S_{1n} & \cdots & T_{1n}S_{1n}\\
T_{21}S_{11} & T_{22}S_{11} & & T_{2n}S_{11} & & T_{21}S_{1n} & T_{22}S_{1n} & & T_{2n}S_{1n}\\
\vdots & & & & & \vdots & & & \\
T_{n1}S_{11} & T_{n2}S_{11} & \cdots & T_{nn}S_{11} & & T_{n1}S_{1n} & T_{n2}S_{1n} & \cdots & T_{nn}S_{1n}\\
\vdots & & & & \cdots & & & \vdots & \\
T_{11}S_{n1} & T_{12}S_{n1} & \cdots & T_{1n}S_{n1} & & T_{11}S_{nn} & T_{12}S_{nn} & \cdots & T_{1n}S_{nn}\\
T_{21}S_{n1} & T_{22}S_{n1} & & T_{2n}S_{n1} & & T_{21}S_{nn} & T_{22}S_{nn} & & T_{2n}S_{nn}\\
\vdots & & & & & \vdots & & & \\
T_{n1}S_{n1} & T_{n2}S_{n1} & \cdots & T_{nn}S_{n1} & & T_{n1}S_{nn} & T_{n2}S_{nn} & \cdots & T_{nn}S_{nn}
\end{pmatrix}\begin{pmatrix} \alpha h_1+\beta k_1\\ \alpha h_2+\beta k_2\\ \vdots\\ \alpha h_{n^2}+\beta k_{n^2}\end{pmatrix}$$

$$= \begin{pmatrix}
T_{11}S_{11}(\alpha h_1+\beta k_1) & T_{12}S_{11}(\alpha h_1+\beta k_1) & \cdots & T_{1n}S_{1n}(\alpha h_1+\beta k_1)\\
T_{21}S_{11}(\alpha h_2+\beta k_2) & T_{22}S_{11}(\alpha h_2+\beta k_2) & & T_{2n}S_{1n}(\alpha h_2+\beta k_2)\\
\vdots & & & \\
T_{n1}S_{11}(\alpha h_n+\beta k_n) & T_{n2}S_{11}(\alpha h_n+\beta k_n) & \cdots & T_{nn}S_{1n}(\alpha h_n+\beta k_n)\\
\vdots & & & \\
T_{n1}S_{n1}(\alpha h_{n^2}+\beta k_{n^2}) & T_{n2}S_{n1}(\alpha h_{n^2}+\beta k_{n^2}) & \cdots & T_{nn}S_{nn}(\alpha h_{n^2}+\beta k_{n^2})
\end{pmatrix}$$

$$= \begin{pmatrix}
T_{11}S_{11}(\alpha h_1)+T_{11}S_{11}(\beta k_1) & \cdots & T_{1n}S_{1n}(\alpha h_1)+T_{1n}S_{1n}(\beta k_1)\\
T_{21}S_{11}(\alpha h_2)+T_{21}S_{11}(\beta k_2) & & T_{2n}S_{1n}(\alpha h_2)+T_{2n}S_{1n}(\beta k_2)\\
\vdots & & \\
T_{n1}S_{11}(\alpha h_n)+T_{n1}S_{11}(\beta k_n) & \cdots & T_{nn}S_{1n}(\alpha h_n)+T_{nn}S_{1n}(\beta k_n)\\
\vdots & & \\
T_{n1}S_{n1}(\alpha h_{n^2})+T_{n1}S_{n1}(\beta k_{n^2}) & \cdots & T_{nn}S_{nn}(\alpha h_{n^2})+T_{nn}S_{nn}(\beta k_{n^2})
\end{pmatrix}$$

8

$$
\begin{aligned}
&= \begin{pmatrix}
T_{11}S_{11}(\alpha h_1) & T_{12}S_{11}(\alpha h_1) & \cdots & T_{1n}S_{1n}(\alpha h_1) \\
T_{21}S_{11}(\alpha h_2) & T_{22}S_{11}(\alpha h_2) & & T_{2n}S_{1n}(\alpha h_2) \\
\vdots & & & \\
T_{n1}S_{11}(\alpha h_n) & T_{n2}S_{11}(\alpha h_n) & \cdots & T_{nn}S_{1n}(\alpha h_n) \\
\vdots & & & \\
T_{n1}S_{n1}(\alpha h_{n^2}) & T_{n2}S_{n1}(\alpha h_{n^2}) & \cdots & T_{nn}S_{nn}(\alpha h_{n^2})
\end{pmatrix} + \\
&\quad \begin{pmatrix}
T_{11}S_{11}(\beta k_1) & T_{12}S_{11}(\beta k_1) & \cdots & T_{1n}S_{1n}(\beta k_1) \\
T_{21}S_{11}(\beta k_2) & T_{22}S_{11}(\beta k_2) & & T_{2n}S_{1n}(\beta k_2) \\
\vdots & & & \\
T_{n1}S_{11}(\beta k_n) & T_{n2}S_{11}(\beta k_n) & \cdots & T_{nn}S_{1n}(\beta k_n) \\
\vdots & & & \\
T_{n1}S_{n1}(\beta k_{n^2}) & T_{n2}S_{n1}(\beta k_{n^2}) & \cdots & T_{nn}S_{nn}(\beta k_{n^2})
\end{pmatrix} \\
&= \alpha \begin{pmatrix}
T_{11}S_{11}(h_1) & T_{12}S_{11}(h_1) & \cdots & T_{1n}S_{1n}(h_1) \\
T_{21}S_{11}(h_2) & T_{22}S_{11}(h_2) & & T_{2n}S_{1n}(h_2) \\
\vdots & & & \\
T_{n1}S_{11}(h_n) & T_{n2}S_{11}(h_n) & \cdots & T_{nn}S_{1n}(h_n) \\
\vdots & & & \\
T_{n1}S_{n1}(h_{n^2}) & T_{n2}S_{n1}(h_{n^2}) & \cdots & T_{nn}S_{nn}(h_{n^2})
\end{pmatrix} + \\
&\quad \beta \begin{pmatrix}
T_{11}S_{11}(k_1) & T_{12}S_{11}(k_1) & \cdots & T_{1n}S_{1n}(k_1) \\
T_{21}S_{11}(k_2) & T_{22}S_{11}(k_2) & & T_{2n}S_{1n}(k_2) \\
\vdots & & & \\
T_{n1}S_{11}(k_n) & T_{n2}S_{11}(k_n) & \cdots & T_{nn}S_{1n}(k_n) \\
\vdots & & & \\
T_{n1}S_{n1}(k_{n^2}) & T_{n2}S_{n1}(k_{n^2}) & \cdots & T_{nn}S_{nn}(k_{n^2})
\end{pmatrix} \\
&= \alpha([T_{ij}] \otimes [S_{ij}])(h) + \beta([T_{ij}] \otimes [S_{ij}])(k)
\end{aligned}
$$

Hence $[T_{ij}] \otimes [S_{ij}]$ is linear. $\square$

Notes and Comments

Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, $T = [T_{ij}] \in M_n(\mathcal{A}) \cong M_n(B(\mathcal{H}))$ and $S = [S_{ij}] \in M_n(\mathcal{B}) \cong M_n(B(\mathcal{K}))$ then,

1. $[T_{ij}] \otimes [S_{ij}] \in M_n(\mathcal{A}) \otimes M_n(\mathcal{B})$ and $\|[T_{ij}] \otimes [S_{ij}]\| = \|[T_{ij}]\|\|[S_{ij}]\|$.
2. $[T_{ij}] \otimes [S_{ij}]$ is a bounded linear operator.

References

[1] **Arveson W.B.**, An Invitation to C^*-Algebras, Springer-Verlag, Berlin 1998 (reprinted).

[2] **Arveson W.B.**: A Short Course on Spectral Theory, Springer-Verlag, Berlin 2002.

[3] **Arveson W.B.**, Subalgebras of C^*-algebras I, Acta Math., **123**(1969), 141-224.

[4] **Blecher D.P. and Paulsen V.I.**, Tensor products of operator spaces, Journal of Functional Analysis. **99**(1991), 262-292.

[5] **Choi M.D.**: A Schwarz inequality for positive linear maps on C^*-algebras, Illinois J. Math. **18**(1974), 565-570.

[6] **Choi M.D.**, Completely positive linear maps on complex matrices, Linear Algebra and Appl., **10**(1975), 285-290.

[7] **Choi M.D.** : Positive linear maps on C^*-algebras, Canad. J. Math., **24**(1972), 520- 529.

[8] **Conway J.B.**: *A Course in Functional Analysis*, Second Edition, Springer-Verlag, Berlin 1990.

[9] **Effros E. G. and Ruan Z.J.**: Operator Spaces, Oxford University Press, Oxford 2000.

[10] **Franka M. B.**, Tensor Products of C^*-algebras, Operator Spaces and Hilbert C^*-modules Math. comm., **4**(1999), 257-268.

[11] **Gheondea A. and Kavruk A.S.**, Absolute continuity of operator valued completely positive maps on C^*-algebras, J. Math. Phys., **50**(2008), 022-102.

[12] **Halmos P. R.** : A Hilbert Space Problem Book. Van Nostrand, Princeton (1967).

[13] **Kubrusly C. S. and Vieira**, Convergence and Decomposition for Tensor Products of Hilbert Space Operators, Math. comm., **3**(2008), 407-416.

[14] **Murphy J. G.,** *C^*-algebras and Operator Theory*, Academic Press Inc., Oval Road, London, 1990.

[15] **Takesaki M.**, *Theory of operator algebras I*, springer, 1979.

[16] **Paulsen V.I.**, A survey of completely bounded maps, University of Houston, Houston, 2007.

[17] **Paulsen V.I.**, Completely Bounded Maps and Operator Algebras, Cambridge studies in advance Mathematics 78, Cambridge University Press, Cambridge, 2003.

[18] **Stinespring W.F.**, Positive functions on C^*-algebras, Proc. Amer. Math. Soc., **6**(1955), 211-216.

