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ON SOME RELATIONS CONNECTING FLUID DYNAMICS AND BI-COMPLEX ANALYSIS

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Abstract: In the paper our main target is to derive some results focusing some connection between fluid dynamics and bi-complex analysis which in fact is the most recent mathematical tool to develop the theory of complex analysis.

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1 INTRODUCTION, DEFINITIONS AND NOTATIONS.

The most recent advancement of the theory of complex numbers lies in the progress of bi-complex analysis. According to Segree [9], a bi-complex number ξ is defined as follows:

$$\xi = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3$$

where x_0, x_1, x_2 and x_3 are real numbers with $i_1^2 = i_2^2 = -1$ and $i_1i_2 = i_2i_1$.
The set of all bi-complex numbers is generally denoted by C_2. In the theory of bi-complex numbers, the sets of real numbers and complex numbers are generally denoted by C_0 and C_1 respectively. Thus
$C_2 = \{\xi : \xi = x_0 + i_1x_1 + i_2x_2 + i_1i_2x_3 \,; x_0, x_1, x_2, x_3 \in C_0\}$.
Equivalently, we may write $C_2 = \{\xi : \xi = z_1 + i_2z_2 \,;\, z_1z_2 \in C_1\}$.
We have seen some illuminating works on the recent advancement of different aspect of bi-complex analysis in Michiji Futagawa [2], E. Hille [3], D. Riley [4], G. Baley Price [1]. In the present paper we would like to establish some results in fluid dynamics in close- connection to bi-complex analysis. In fact, the paper is an improved version of Datta and Sen [15] and therefore all the preliminary theories and definitions on bi-complex analysis as required here are omitted.

Now, let us define a function as follows:

Let $\Psi: [0,\infty) \to (0,\infty)$ be a non-decreasing unbounded function, satisfying the following two conditions:

$$(i)\ \lim_{r\to\infty} \frac{\log^{[p]}(r)}{\log^{[q]}[\Psi(r)]} = 1$$

and

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[q]}(\alpha r)}{\log^{[q]}\Psi[(r)]} = 1,$$

for some α>1 and p, q are any two posotive integers.

With the help of function Ψ as defined earlier the following definitions may be given :

Definition 1. The Ψ −order $\rho_{(F,\Psi)}$ of a bicomplex meromorphic function $F(w) = F_{e_1}(z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2$

is defined as $$\rho_{(F,\Psi)} = max\left\{\rho_{(F_{e1},\Psi)}, \rho_{(F_{e2},\Psi)}\right\}$$

where

$$\rho_{F_{e_i},\Psi} = \limsup_{r_i\to\infty} \frac{loglogM_i\left(r_i, F_{e_i}\right)}{log[\Psi(r_i)]} for\ i = 1,2$$

Remark 1. The Ψ −lower order $\lambda_{(F,\Psi)}$of a bicomplex meromorphic function is defined as

$$\lambda_{(F,\Psi)} = min\left\{\lambda_{(F_{e1},\Psi)}, \lambda_{(F_{e2},\Psi)}\right\},$$

where

$$\lambda_{(F_{e_i},\Psi)} = \liminf_{r_i\to\infty} \frac{loglogM_i\left(r_i, F_{e_i}\right)}{log[\Psi(r_i)]}\ for\ i = 1,2.$$

Remark 2. The Ψ −hyper order $\bar{\rho}_{(F,\Psi)}$(Ψ −hyper lower order $\bar{\lambda}_{(F,\Psi)}$) and the generalized Ψ -order $\rho^{(k)}_{(F,\Psi)}$ (generalized Ψ -lower order $\lambda^{(k)}_{(F,\Psi)}$) can also be defined in a similar way.

Definition 2. The Ψ -type of F $\sigma_{(F,\Psi)}$ of a bicomplex meromorphic function is defined as

$$\sigma_{(F,\Psi)} = max\left\{\sigma_{(F_{e1},\Psi)}, \sigma_{(F_{e2},\Psi)}\right\}$$

where

$$\sigma_{(F_{e_i},\Psi)} = \limsup_{r_i\to\infty} \frac{log\, M_i\left(r_i, F_{e_i},\Psi\right)}{\Psi(r_i^{\ \sigma(F_{e_i},\Psi)})}\ and\ 0 < \rho_{(F_{e_i},\Psi)} < \infty\ \ for\ i = 1,2.$$

Definition 3. Let $F(w)$ be an entire function of Ψ −order zero. Then the quantities $\rho^*_{(F,\Psi)}$ and $\lambda^*_{(F,\Psi)}$ can be defined as

$$\rho^*_{(F,\Psi)} = max\left\{\rho^*_{(F_{e_1},\Psi)}, \rho^*_{(F_{e_2},\Psi)}\right\}$$

$$and \qquad \lambda^*_{(F,\Psi)} = min\left\{\lambda^*_{(F_{e_1},\Psi)}, \lambda^*_{(F_{e_2},\Psi)}\right\}$$

$$\text{where } \rho^*_{(F_{e_i},\Psi)} = \limsup_{r_i\to\infty} \frac{loglogM_i\left(r_i, F_{e_i}\right)}{loglog[\Psi(r_i)]}$$

$$and \qquad \lambda^*_{F_{e_i}} = \liminf_{r_i\to\infty} \frac{loglogM_i\left(r_i, F_{e_i}\right)}{loglog[\Psi(r_i)]}\ for\ \ i = 1,2.$$

Definition 4. Let *F(w)* be an entire function order zero. Then the quantities $\rho^{**}_{(F,\Psi)}$and $\lambda^{**}_{(F,\Psi)}$ can be defined as

$$\rho^{**}_{(F,\Psi)} = max\left\{\rho^{**}_{(F_{e_1},\Psi)}, \rho^{**}_{(F_{e_2},\Psi)}\right\} \ and$$

$$\lambda^{**}_{(F,\Psi)} = min\left\{\lambda^{**}_{(F_{e_1},\Psi)}, \lambda^{**}_{(F_{e_2},\Psi)}\right\} \text{ where}$$

$$\rho^{**}_{F_{e_i}} = \limsup_{r_i\to\infty} \frac{logM_i(r_i, F_{e_i})}{log[\Psi(r_i)]} \ and \ \lambda^{**}_{F_{e_i}} = \liminf_{r_i\to\infty} \frac{logM_i(r_i, F_{e_i})}{log[\Psi(r_i)]} \ for\ i = 1,2.$$

Definition 5. (Factorization of F(w)) Let F(w) be a bicomplex meromorphic function on $T \subset C_2$. Then F is said to have f and g as left and right factors respectively if F_{ei} has f_{ei} and g_{ei} as left and right factors respectively for i=1,2, i.e., f_{ei} is meromorphic and g_{ei} is entire for i=1,2.

Definition 6. (Complex potential flow) If $f(z) = u(x,y) + iv(x,y) \in C_1$ be a complex function where $u(x,y) \in R^2$ and $v(x,y) \in R^2$satisfy the Cauchy-Riemann equations, i.e.,$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and Laplace's equation, i.e., $\frac{\partial^2_u}{\partial x^2} + \frac{\partial^2_u}{\partial y^2} = 0$, then $f(z)$ can be termed as a complex potential fluid flow.

Definition 7.(Bicomplex potential fluid flow) If $f(w) = f(z_1 + i_2z_2) = f_{e_1}(z_1 - i_1z_2)e_1 + f_{e_2}(z_1 +$ $i1z2e2$ be the idempotent composition of two complex functions, with $f_{e_1}(z_1 - i_1z_2) = u(z_1, z_2) - i_1v(z_1 - z_2) \in C_1$ and $f_{e_2}(z_1 - i_1z_2) = u(z_1, z_2) + i_1v(z_1, z_2) \in C_1$ where $u(z_1, z_2)$ and $v(z_1, z_2)$ satisfy Cauchy-Riemann equations and Laplace's equation, i.e.,$\frac{\partial u}{\partial z_1} = \frac{\partial v}{\partial z_2}, \frac{\partial u}{\partial z_2} = -\frac{\partial v}{\partial z_2}$and $\frac{\partial^2_u}{\partial z_1^2} + \frac{\partial^2_v}{\partial z_2^2} = 0, \frac{\partial^2_u}{\partial z_1^2} + \frac{\partial^2_v}{\partial z_2^2} = 0$, therefore $f_{e_1}(z_1 - i_1z_2)$ and $f_{e_2}(z_1 + i_1z_2)$ can be termed as complex potential fluid flows. So, $f(w)$ can be termed as a composition of two different potential fluid flows f_{e_1} and f_{e_2}.

3 LEMMA.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 [8] [15]. If $f(z) = u(x,y) + iv(x,y)$ be complex potential fluid flow defined in the region $\{y > 0\}$ satisfying the following properties :

(i) $f(z)$ is continuously differentiable in the region $\{y \geq 0\}$,
(ii) $f'(z)$ is parallel to the x-axis when $y = 0$ and
(iii) $f'(z)$ is uniformly bounded in $\{y > 0\}$, then the order and lower order of $f(z)$ are zero.

Corollary 1. If $f(w) = f(z_1 + i_2z_2) = f_{e_1}(z_1 - i_1z_2)e_1 + f_{e_2}(z_1 + i_1z_2)e_2$ be an idempotent composition of two complex potential fluid flows satisfying the following properties :

(i) f_{e_1} and f_{e_2} are continuously differentiable in the region $\{y \geq 0\}$
(ii) f'_{e_1} and f'_{e_2} are parallel to the x-axis when $y = 0$ and
(iii) f'_{e_1} and f'_{e_2} are uniformly bounded in $\{y > 0\}$, then the order and lower order of $f(w)$ are zero.

Lemma 2 [14]. If $f(z)$ and $g(z)$ are any two entire functions , then for all sufficiently large values of r,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0), f|\right) \leq M(r, fog) \leq M(M(r,g), f).$$

Lemma 3[14]. If f be entire and g be a transcendental entire function of finite lower order, then for any $\delta > 0$,

$$M(r^{1+\delta}, fog) \geq M(M(r,g), f). \qquad (r \geq r_0)$$

Lemma 4 [13]. If F has f and $g$as left and right factors, then we always have the following factorization :

$$F(w) = f\big(g(w)\big).$$

4 THEOREMS.

In this section we present our main results of the paper.

Theorem 1. If $f(w) = f(z_1 + i_2 z_2) = f_{e_1}(z_1 - i_1 z_2)e_1 + f_{e_2}(z_1 + i_1 z_2)e_2$ be an idempotent composition of two complex potential fluid flows f_{e_1} and f_{e_2} satisfying the following properties :

(i) f_{e_1} and f_{e_2} are continuously differentiable in the region $\{y \geq 0\}$

(ii) $f_{e_1}^{'}$ and $f_{e_2}^{'}$ are parallel to x-axis when $y = 0$ and

(iii) $f_{e_1}^{'}$ and $f_{e_2}^{'}$ are uniformly bounded in $\{y > 0\}$,

$$then\ \rho_f^{'} = 1\ and\ \lambda_f^{'} = 1.$$

Proof. From the definitions of $\rho_{f,\Psi}^{**}$and $\lambda_{f,\Psi}^{**}$ and using Definition 4, we have for arbitrary positive $\varepsilon_1, \varepsilon_2$ and all sufficiently large values of r_1,r_2

$$logM_1\big(r_1, f_{e_1}\big) \leq \Big(\rho_{(f_{e_1},\Psi)}^{**} + \varepsilon_1\Big)\log[\Psi(r_1)]\text{and}$$

$$logM_2\big(r_2, f_{e_2}\big) \leq \Big(\rho_{(f_{e_2},\Psi)}^{**} + \varepsilon_2\Big)\, log[\Psi(r_2)]$$

$$\text{Therefore, } loglogM_1\big(r_1, f_{e_1}\big) \leq loglog[\Psi(r_1)] + O(1)$$

$$\text{and} \quad loglogM_2\big(r_2, f_{e_2}\big) \leq loglog[\Psi(r_2)] + O(1)$$

$$\text{i.e.,} \frac{loglog\ M_1\left(r_1, f_{e_1}\right)}{\text{loglog}\ [\Psi(r_1)]} \leq \frac{\text{loglog}\ [\Psi(r_1)] + O(1)}{\text{loglog}\ [\Psi(r_1)]}\ and$$

$$\frac{loglogM_2\big(r_2, f_{e_2}\big)}{loglogr_2} \leq \frac{loglog[\Psi(r_2)] + O(1)}{loglog[\Psi(r_2)]}$$

$$i.e., \limsup_{r_1\to\infty} \frac{loglogM_1\big(r_1, f_{e_1}\big)}{loglog[\Psi(r_1)]} \leq 1\ and\ \limsup_{r_2\to\infty} \frac{loglogM_2\big(r_2, f_{e_2}\big)}{loglog[\Psi(r_2)]} \leq 1$$

i.e.,$\rho_{f_{e_1},\Psi}^{*} \leq 1$ and $\rho_{f_{e_2},\Psi}^{*} \leq 1$ and therefore using Definition 3,

$$we\ have\ \rho_{f,\Psi}^{*} \leq 1. \qquad (1)$$

Similarly, proceeding as above and using Definition 3,

$$we\ have\ \lambda_{f,\Psi}^{*} \leq 1. \qquad (2)$$

Again, for arbitrary positive ε_1,ε_2 and all sufficiently large values of r_1,r_2 we have

$$logM_1\big(r_1, f_{e_1}\big) \geq \Big(\lambda_{(f_{e_1},\Psi)}^{**} - \varepsilon_1\Big)\log[\Psi(r_1)]$$

and$logM_2\big(r_2, f_{e_2}\big) \geq \Big(\lambda_{(f_{e_2},\Psi)}^{**} - \varepsilon_2\Big)\, log[\Psi(r_2)]$

Therefore,$loglogM_1\big(r_1, f_{e_1}\big) \geq loglog[\Psi(r_1)] + O(1)$

and$loglog M_2(r_2, f_{e_2}) \geq loglog[\Psi(r_2)] + O(1)$

$$\text{i.e.,} \frac{loglog\ M_1(r_1, f_{e_1})}{\text{loglog}\ [\Psi(r_1)]} \geq \frac{\text{loglog}\ [\Psi(r_1)] + O(1)}{\text{loglog}\ [\Psi(r_1)]}$$

and

$$\frac{loglog M_2(r_2, f_{e_2})}{loglog[\Psi(r_2)]} \geq \frac{loglog[\Psi(r_2)] + O(1)}{loglog[\Psi(r_2)]}$$

$$i.e., \limsup_{r_1\to\infty} \frac{loglog M_1(r_1, f_{e_1})}{\text{loglog}[\Psi(r_1)]} \geq 1$$

$$and \limsup_{r_2\to\infty} \frac{loglog M_2(r_2, f_{e_2})}{loglog[\Psi(r_2)]} \geq 1$$

i.e.,$\rho^*_{(f_{e_1}, \Psi)} \geq 1$ and $\rho^*_{(f_{e_2}, \Psi)} \geq 1$ and therefore using Definition 3,

we have $\rho^*_{f,\Psi} \geq 1$. (3)

Similarly, proceeding as above and using Definition 3,

we have $\lambda^*_{f,\Psi} \geq 1$. (4)

From (1) and (3) we have $\rho^*_{f,\Psi} = 1$ and from (2) and (4) we have $\lambda^*_{f,\Psi} = 1$.

Thus the theorem follows.

Example 1. Let $f(w) = w = z_1 + i_2 z_2 = (z_1 - i_1 z_2)e_1 + (z_1 + i_1 z_2)e_2 \in C_2$ (bicomplex space) be a bicomplex potential fluid flow in C_2.

Therefore, $f_{e_1} = z_1 - i_1 z_2 \in C_1$ (or C) and $f_{e_2} = z_1 + i_1 z_2 \in C_1$(or C)

$$Therefore,\ \rho_{f_{e_1}, \Psi} = \limsup_{r_1\to\infty} \frac{loglog M_1(r_1, f_{e_1})}{log[\Psi(r_1)]} = 0\ \ as\ M_1(r_1, f_{e_1}) \leq |z_1 - i_1 z_2| \leq r_1 + r_2$$

$$and\ \rho_{(f_{e_2}, \Psi)} = \limsup_{r_2\to\infty} \frac{loglog M_2(r_2, f_{e_2})}{\text{log}[\Psi(r_2)]} = 0\ as\ M_2(r_2, f_{e_2}) \leq |z_1 + i_1 z_2| \leq r_1 + r_2$$

Hence $\rho^*_{f,\Psi} = 0$ and similarly $\lambda^*_{f,\Psi} = 0$.

$$\rho^*_{(f_{e_1}, \Psi)} = \limsup_{r_1\to\infty} \frac{loglog M_1(r_1, f_{e_1})}{loglog[\Psi(r_1)]} = \limsup_{r_2\to\infty} \frac{loglog(r_1 + r_2)}{loglog[\Psi(r_1)]} = 1, \quad r_1 \text{is fixed.}$$

$$And\ \rho^*_{(f_{e_2}, \Psi)} = \limsup_{r_2\to\infty} \frac{loglog M_2(r_2, f_{e_2})}{loglog[\Psi(r_2)]} = \limsup_{r_2\to\infty} \frac{loglog(r_1 + r_2)}{loglog[\Psi(r_2)]} = 1, r_2 \text{ is fixed.}$$

Therefore,$\rho^*_{f,\Psi} = 1$ and similarly $\lambda^*_{f,\Psi}$=1.

Similarly, we can also show that $\rho^{**}_{f,\Psi} = 1$ and$\lambda^{**}_{f,\Psi} = 1$.

Theorem 2. Let *f(w)* and *g(w)* be any two bicomplex potential fluid flows satisfying the following properties :

(i) f_{e_1}, f_{e_2} are continuously differentiable in the region $\{y \geq 0\}$
(ii) f'_{e_1}, f'_{e_2} are parallel to the *x*-axis when $y = 0$
(iii) f'_{e_1}, f'_{e_2} are uniformly bounded in $\{y > 0\}$, such that $\rho_f = 0$ and $\lambda_f < \infty$.

Then $\rho_{f,\Psi} = \rho_{g,\Psi}$.

Proof. Using Lemma 4, we can say that *F(w)* can be factorized to *f(g(w))*. Now, using Lemma 2 and Theorem 1 we have

$$\rho_{(F_{e_1},\Psi)} = \rho_{\{(f_{e_1} o g_{e_1}),\Psi\}} = \limsup_{r_1\to\infty} \frac{loglogM_1\left(r_1, f_{e_1} o g_{e_1}\right)}{log[\Psi(r_1)]}$$

$$\leq \limsup_{r_1\to\infty} \frac{loglogM\left(M\left(r_1, g_{e_1}\right), f_{e_1}\right)}{loglogM(r_1, g)} . \limsup_{r_1\to\infty} \frac{loglogM\left(r_1, g_{e_1}\right)}{log[\Psi(r_1)]}$$

$$\leq \rho^*_{(f_{e_1},\Psi)} . \rho_{(g_{e_1},\Psi)} = 1. \rho_{(g_{e_1},\Psi)} = \rho_{g(e_1,\Psi)}$$

Similarly,

$$\rho_{(F_{e_2},\Psi)} \leq \rho_{(g_{e_2},\Psi)}.$$

Therefore

$$\rho_{F,\Psi} = \rho_{\{(fog),\Psi\}} \leq max\left\{\rho_{(g_{e_1},\Psi)}, \rho_{(g_{e_2},\Psi)}\right\} = \rho_{g,\Psi},$$

$$\text{i.e.,} \rho_{F,\Psi} \leq \rho_{g,\Psi}. \tag{5}$$

Now using Lemma 3 and Theorem 1, we have

$$\rho_{(F_{e_1},\Psi)} = \rho_{\{(f_{e_1} o g_{e_1}),\Psi\}} = \limsup_{r_1\to\infty} \frac{loglogM_1\left(r_1^{1+\delta}, f_{e_1} o g_{e_1}\right)}{\log[\Psi({r_1}^{1+\delta})}$$

$$\geq \liminf_{r_1\to\infty} \frac{loglogM\left(M\left(r_1, g_{e_1}\right), f_{e_1}\right)}{loglogM\left(r_1, g_{e_1}\right)} . \limsup_{r_1\to\infty} \frac{loglogM\left(r_1, g_{e_1}\right)}{\log[\Psi(r_1)]}$$

$$\geq \lambda^*_{(f_{e_1},\Psi)} . \rho_{(g_{e_1},\Psi)} = 1. \rho_{(g_{e_1},\Psi)} = \rho_{(g_{e_1},\Psi)}$$

Similarly, $\rho_{(F_{e_2},\Psi)} \geq \rho_{(g_{e_2},\Psi)}$.

Therefore

$$\rho_{F,\Psi} = \rho_{\{(fog),\Psi\}} \geq max\left\{\rho_{(g_{e_1},\Psi)}, \rho_{(g_{e_2},\Psi)}\right\} = \rho_{g,\Psi},$$

$$\text{i.e.,} \ \rho_{F,\Psi} \geq \rho_{g,\Psi}. \tag{6}$$

Therefore from (5) and (6), the result follows.

Theorem 3. Let *f(w)* and *g(w)* be any two bicomplex potential fluid flows satisfying the following properties :

(i) *f(w)* is entire and *g(w)* is transcendental such that $\rho_{fog} = 0$ *and* $\lambda_g < \infty$.

Then $\rho^{**}_{f,\Psi}\lambda^{**}_{g,\Psi} \leq \rho^{**}_{(fog),\Psi} \leq \rho^{**}_{f,\Psi}\rho^{**}_{g,\Psi}$.

Proof. Using Lemma 3 we have

$$\rho^{**}_{(F_{e1},\Psi)} = \rho^{**}_{\{(f_{e_1}og_{e_1}),\Psi\}} = \limsup_{r_1\to\infty} \frac{logM_1(r_1^{1+\delta}, f_{e_1}og_{e_1})}{log[\Psi(r_1^{1+\delta})}$$

$$\geq \limsup_{r_1\to\infty} \frac{logM_1(M_1(r_1, g_{e_1}), f_{e_1})}{logM_1(r_1, g)} . \liminf_{r_1\to\infty} \frac{logM_1(r_1, g_{e_1})}{\log[\Psi(r_1)}$$

$$= \rho^{**}_{f_{e_1}}\lambda^{**}_{g_{e_1}}.$$

Similarly,

$$\rho^{**}_{(F_{e_2},\Psi)} = \rho^{**}_{\{(f_{e_2}og_{e_2}),\Psi\}} \geq \rho^{**}_{(f_{e_2},\Psi)}.\lambda^{**}_{(g_{e_2},\Psi)}.$$

Therefore

$$\rho^{**}_{fog,\Psi} = max\left\{\rho^{**}_{\{(f_{e_1,\Psi})o(g_{e_1},\Psi)\}}, \rho^{**}_{\{(f_{e_2},\Psi)o(g_{e_2},\Psi)\}}\right\} \geq \rho^{**}_{f,\Psi}.\lambda^{**}_{g,\Psi} \qquad (7)$$

Again, by using Lemma 2 we have

$$\rho^{**}_{(F_{e_1},\Psi)} = \rho^{**}_{f_{e_1}og_{e_1},\Psi} = \limsup_{r_1\to\infty} \frac{logM_1(r_1^{1+\delta}, f_{e_1}og_{e_1})}{\log[\Psi(r_1^{1+\delta})}$$

$$\leq \limsup_{r_1\to\infty} \frac{logM_1(M_1(r_1, g_{e_1}), f_{e_1})}{logM_1(r_1, g)} . \limsup_{r_1\to\infty} \frac{logM_1(r_1, g_{e_1})}{log[\Psi(r_1)}$$

$$= \rho^{**}_{f_{e_1}}\rho^{**}_{g_{e_1}}.$$

Similarly,

$$\rho^{**}_{(F_{e_2},\Psi)} = \rho^{**}_{\{(f_{e_2}og_{e_2}),\Psi\}} \leq \rho^{**}_{(f_{e_2},\Psi)}.\rho^{**}_{(g_{e_2},\Psi)}.$$

Therefore

$$\rho^{**}_{(fog),\Psi} = max\left\{\rho^{**}_{(f_{e_1},\Psi)o(g_{e_1},\Psi)}, \rho^{**}_{(f_{e_2},\Psi)o(g_{e_2},\Psi)}\right\} \leq \rho^{**}_{f,\Psi}\rho^{**}_{g,\Psi}. \quad (8)$$

Therefore, from (7) and (8) the result follows.

CONCLUSION AND FUTURE PROSPECTS:

The estimates and derivation carried out in the paper may also be established under the treatment of bicomplex valued functions of slower and faster growth.

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