

PROBABILISTIC (Q,r) INVENTORY MODEL WITH PARTIALLY BACKORDERS WHEN LEAD TIME DEMAND NORMALLY DISTRIBUTED UNDER HOLDING COST RESTRICTION

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Abstract: This paper derives the probabilistic continuous review inventory model that has the two types of shortage when the order cost is a function of the order quantity. The objective is to minimize the expected annual total cost under a constraint on the expected holding cost when the lead time demand follows Normal distribution by using the Lagrangian method. Some published special cases are deduced and a numerical application with illustrative graphs is added.

Key words: Probabilistic model, partially backorders, continuous review inventory system, safety stock, varying order cost, lead-time demand, Normal distribution.

1. INTRODUCTION

The two basic questions that any inventory control system has to answer are when and how much to order. Over the years, hundreds of papers and books have been published presenting models for doing this under a wide variety of conditions and assumptions. Most of authors have shown that if demand that cannot be filled from stock is backordered or using the lost sales model. Rabinowitz et al. [4] modeled a $\langle Q, r \rangle$ inventory system using a control variable, which limits the maximum number of backorders allowed to accumulate during a cycle. Also, Zipkin [11] shows that if demands occurring during a stockout period are lost sales rather than backorders, the optimal policy is to have either no stockout or all stockouts.

Several $\langle Q, r \rangle$ inventory models with mixture of backorders and lost sales were proposed by Posner and yansouni [10], Montgomery et al. [1], Rosenberg [2], and Park [8]. Almost all the previous research works used γ as a fraction of unsatisfied demand that will be backordered (the remaining fraction $(1 - \gamma)$ completely lost) to model partial backorders. Since it is optimal to allow some stockouts if all customers will wait ($\gamma = 1$) and it is optimal to either allow no stockouts or lose all sales if customers have no patience ($\gamma = 0$).

Recently, El-Wakeel and Fergany [9] deduced Constrained Probabilistic Continuous Review Inventory System with Mixture Shortage and Stochastic Lead Time Demand. Fergany and El-Sodany [7] derived probabilistic periodic review backorders and lost sales models under constraint and varying holding cost and normally distributed protection interval demand. Also, El-wakeel and Fergany [5,6] introduced probabilistic lost sales models with normal distribution and other continuous distribution.

In this study, we assume that both backorder and lost sales costs are independent of the duration of the stockout and γ is the backorder fraction. Also, we deduced the model with varying order cost when the demand is a random variable, the lead-time is constant and the lead time demand is normally distributed under the holding cost constraint. The situation will be considered in which a single item is stocked to meet a probabilistic demand.

2. NOTATIONS AND ASSUMPTIONS

The following notations are adopted for developing our model:

- $\bar{D}$ = The average rate of annual demand ,
 Q = A decision variable representing the order quantity per cycle,
 r = A decision variable representing the reorder point,
 N = The inventory cycle,
 n = The average number of cycles per year,
 L = The lead – time between the placement of an order and its receipt,
 γ = a fraction of unsatisfied demand that will backordered,
 x = The continuous random variable represents the demand during L (lead – time demand),
 $f(x)$ = The probability density function of the lead – time demand and $F(x)$ its distribution function,
 $r - x$ = The random variable represents the net inventory when the procurement quantity arrives if the lead - time demand $x \leq r$,

$$E(r - x) = ss = \text{Safety stock} = \text{The expected net inventory} = \int_0^r (r - x) f(x) dx = r - E(x) + \int_r^\infty (x - r) f(x) dx$$

$$\begin{aligned} \bar{H} &= \text{The average on hand inventory} \\ &= \frac{\text{Max. on hand} + \text{Min. on hand}}{2} = \frac{ss + Q + ss}{2} = \frac{Q}{2} + r - E(x) + \int_r^\infty (x - r) f(x) dx \end{aligned}$$

$$R(r) = \text{The reliability function} = 1 - F(r) = \int_r^\infty f(x) dx,$$

$$\bar{S}(r) = \text{The expected value of shortages per cycle} = \int_r^\infty (x - r) f(x) dx,$$

c_b = The backorders cost per cycle,

c_o = The order cost per cycle,

$C_o(Q) = c_o Q^\beta$ = The varying order cost per cycle, $0 < \beta < 1$, where β is a constant real number selected to provide the best fit of estimated expected cost function.

c_h = The holding cost per year,

c_l = The lost sales cost per cycle,

K = The limitation on the expected annual holding cost.

THE FOLLOWING ASSUMPTIONS ARE ADOPTED FOR DEVELOPING OUR MODEL:

The system is a continuous review which means that the inventory levels are reviewed continuously and orders are recorded and then the inventory level is known at all times. An order quantity of size Q per cycle is placed every time the inventory level reaches a certain reorder point r . The problem is to determine the optimal values of Q and r which minimize the expected annual total cost. Thus, the following assumptions are made for developing the mathematical model:

- 1) The cycle N is defined as the time between two successive arrivals of orders $N = \frac{Q}{\bar{D}}$
- 2) Assume that the system repeats itself in the sense that the inventory position varies between r and $r + Q$ during each cycle.

- 3) The average number of cycles per year can be written as $n = \frac{1}{N}$.
- 4) There is never more than a single order outstanding.
- 5) When the number of units on hand and on order reaches the reorder point r , action is initiated to procure a replenishment quantity Q .
- 6)

3. THE MATHEMATICAL MODEL

In this model, the expected annual total cost consisted of the sum of three components: the expected varying order cost, the expected holding cost and the expected shortage cost as follows:

$$E(TC) = E(OC) + E(HC) + E(BC) + E(LC) \quad (1)$$

Where $E(OC) = c_o Q^\beta \frac{\bar{D}}{Q} = c_o \bar{D} Q^{\beta-1}$ (2)

$$E(HC) = c_h \left(\frac{Q}{2} + r - E(x) + (1-\gamma) \int_r^\infty (x-r) f(x) dx \right) \quad (3)$$

$$E(BC) = \frac{c_b \gamma \bar{D}}{Q} \int_r^\infty (x-r) f(x) dx \quad (4)$$

and $E(LC) = \frac{c_l (1-\gamma) \bar{D}}{Q} \int_r^\infty (x-r) f(x) dx$ (5)

Therefore

$$E[TC(Q, r)] = c_o \bar{D} Q^{\beta-1} + c_h \left(\frac{Q}{2} + r - E(x) \right) + \frac{c_b \gamma \bar{D}}{Q} \int_r^\infty (x-r) f(x) dx + \left(\frac{c_l \bar{D}}{Q} + c_h \right) (1-\gamma) \int_r^\infty (x-r) f(x) dx \quad (6)$$

Our objective is to minimize the expected annual total cost $E[TC(Q, r)]$ under the expected holding cost constraint:

$$c_h \left(\frac{Q}{2} + r - E(x) + (1-\gamma) \int_r^\infty (x-r) f(x) dx \right) \leq K \quad (7)$$

To solve this primal function which is a convex programming problem, let us write it in the following form:

$$E[TC(Q, r)] = c_o \bar{D} Q^{\beta-1} + c_h \left(\frac{Q}{2} + r - E(x) \right) + \frac{c_b \gamma \bar{D}}{Q} \bar{S}(r) + \left(\frac{c_l \bar{D}}{Q} + c_h \right) (1-\gamma) \bar{S}(r) \quad (8)$$

$$\text{Subject to: } c_h \left(\frac{Q}{2} + r - E(x) + (1-\gamma) \bar{S}(r) \right) \leq K \quad (9)$$

To find the optimal values Q^* and r^* which minimize equation (8) under the constraint (9), we will use the Lagrange multiplier technique as follows:

$$L(Q, r, \lambda) = c_o \bar{D} Q^{\beta-1} + c_h \left(\frac{Q}{2} + r - E(x) \right) + \frac{c_b \gamma \bar{D}}{Q} \bar{S}(r) + \left(\frac{c_l \bar{D}}{Q} + c_h \right) (1-\gamma) \bar{S}(r) + \lambda \left[c_h \left(\frac{Q}{2} + r - E(x) + (1-\gamma) \bar{S}(r) \right) - K \right] \quad (10)$$

Where λ is the Lagrange multiplier.

The optimal values Q^* and r^* can be found by setting each of the corresponding first partial derivatives of equation (10) equal to zero, then we obtain:

$$A Q^2 - B Q^\beta - 2[\gamma(M - G) + G] \bar{S}(r) = 0 \quad (11)$$

$$\text{and } R(r) = \frac{A Q}{\gamma M + (1 - \gamma)(G + A Q)} \quad (12)$$

Where $A = (1 + \lambda)c_h$, $B = 2(1 - \beta)c_o \bar{D}$, $G = c_l \bar{D}$ and $M = c_b \bar{D}$.

Clearly there is no closed form solution of equations (11) and (12).

4. LEAD-TIME DEMAND FOLLOWS NORMAL DISTRIBUTION

Assume that the lead-time demand follows the Normal distribution. So we can minimize the expected annual total cost mathematically as follows:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}; \quad x \geq 0, \quad \text{with } E(x) = \mu, \text{ var}(x) = \sigma^2$$

$$R(r) = \Phi(Z), \quad \text{where } Z = \frac{r - \mu}{\sigma}$$

$$\text{and } \bar{S}(r) = (\mu - r) \Phi(Z) + \sigma \phi(Z) \quad (13)$$

$$\text{where } \phi(Z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} \quad \text{and} \quad \Phi(Z) = \int_z^\infty \phi(x) dx$$

Substituting form (13) into (11) and (12), we get:

$$A Q^{*2} - B Q^{*\beta} - 2[\gamma(M - G) + G] [(\mu - r) \Phi(Z) + \sigma \phi(Z)] = 0 \quad (14)$$

$$\text{and } \Phi(Z) = \frac{A Q^*}{\gamma M + (1 - \gamma)(G + A Q^*)} \quad (15)$$

Also, solving the pairs of equations (14) and (15), we have to use the iterative method.

5. SPECIAL CASES

Case 1: When $\gamma = 0$, $\beta = 0$ and $K \rightarrow \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$. Thus equations (14) and (15) become:

$$Q^* = \sqrt{\frac{2\bar{D}[c_o + c_l \bar{S}(r)]}{c_h}} \quad \text{and} \quad \Phi\left(\frac{r^* - \mu}{\sigma}\right) = \left(\frac{c_h Q^*}{c_l \bar{D} + c_h Q^*}\right) \quad \text{This is unconstrained lost}$$

sales continuous review inventory model with lead-time demand follows the Normal distribution and constant order cost as given by Hadley [3].

Case 2: Let $\gamma = 0$, $\beta = 0$ and $K \rightarrow \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$. Thus equations (11) and (12) become:

$$Q^* = \sqrt{\frac{2\bar{D}(c_o + c_l \bar{S}(r))}{c_h}} \quad \text{and} \quad R(r^*) = \frac{c_h Q^*}{c_h Q^* + c_l \bar{D}}$$

This is unconstrained lost sales continuous review inventory model with constant units of cost, which are the same results as in Hadley [3].

Case 3: Let $\gamma = 1$, $\beta = 0$ and $K \rightarrow \infty \Rightarrow C_o(Q) = c_o$ and $\lambda = 0$. Thus, equations (11) and (12) become:

β	Q^*	r^*	minE(TC)
0.1	1561	191.2	17123.5
0.2	1580	184.7	26350.5
0.3	1594	177.6	45525.5
0.4	1609	170	85429.9
0.5	1625	162	168498.2
0.6	1641	153.6	342001.5
0.7	1657	145	704881.4
0.8	1670	137.7	1465291.4
0.9	1673	136.5	3061493

$$Q^* = \sqrt{\frac{2\bar{D}(c_o + c_b \bar{S}(r))}{c_h}} \quad \text{And} \quad R(r^*) = \frac{c_h}{c_b \bar{D}} Q^*$$

This is unconstrained backorders continuous review inventory model with constant unit costs, which are the same results as in Hadley [3] and Fabrycky & Banks [13].

6. AN APPLICATION

The cosmetics department of a large department store has recently introduced a constrained $\langle Q, r \rangle$ system with varying order cost and mixed shortages to control many items in the department. A particular type of expensive perfume has an annual demand rate equals 1600 units. The cost of placing an order amounts to \$4000 and the inventory holding cost is \$10. This particular perfume is not easy to obtain elsewhere, and hence demands occurring when the store is out of stock are partially backordered. The management estimates that 70% of unsatisfied demand will be backordered with backorder cost equals \$600 and the remaining demand will be lost with cost \$2000. There is a restriction that the average holding cost is either less than or equal \$8500 per year and the procurement lead-time is constant. Determine Q^* and r^* when the lead time demand has Normal distribution with $\mu = 125$ and $\sigma = 20\sqrt{2}$ units.

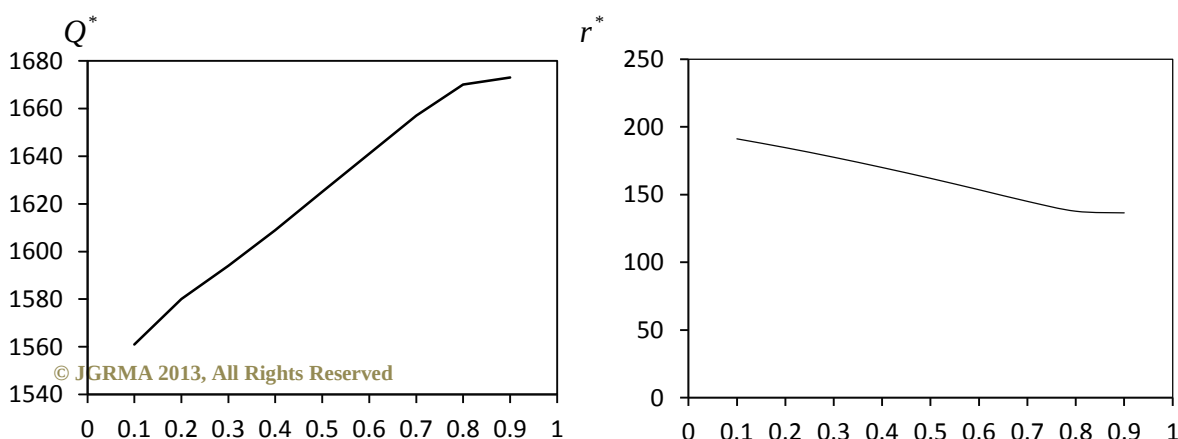
We have the parameters values:

$$\bar{D} = 1600, c_o = \$4000, c_h = \$10, c_b = \$600, \gamma = 0.7, c_l = \$2000 \text{ and } K = \$8500.$$

By solving the previous deduced equations at different values of β , we obtain:

Table 1: The optimal solutions and the min E(TC) for each distribution at $\gamma = 0.7$

From the data given in Table 1, we can draw figures of the optimal values of Q^* , r^* and minE(Tc) against β as in the following Figures (1), (2) and (3):

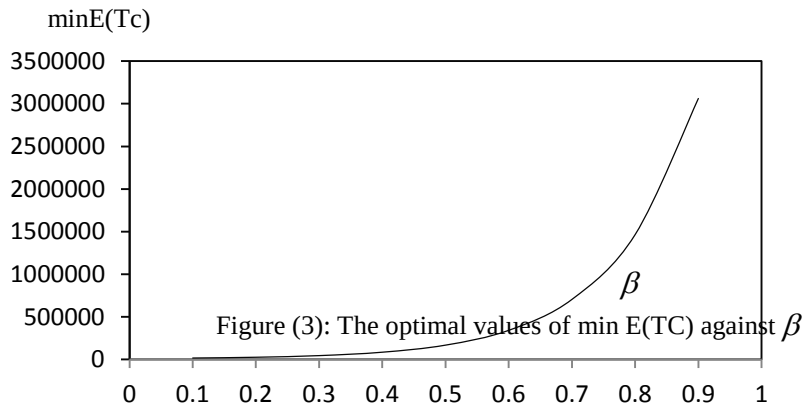


β

β

Figure (1): The optimal values of Q^* against β

Figure (2): The optimal values of r^* against β



7. CONCLUSION

This section deducing our probabilistic (Q,r) model with partially lost of customers when lead-time demand follows Normal distribution. For such distribution, we can evaluate the solution of Q^* and r^* for each value of β and λ which holds our constraint on the expected holding cost and then obtain the minimum expected total cost. From the previous example, we can deduce that the optimal minE(TC) when the lead-time demand follows Normal distribution will be at $\beta = 0.1$. Also, we draw the curves of Q^* , r^* and minE(TC) against β for the model, which indicate the value of β that minimizes the expected annual total cost of our application.

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