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On The Maximal Ideals In The Banach Space Of $^{-}$Quasicontinuous Functions

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Abstract: In this paper, some interesting properties of $^{-}$Quasicontinuous functions are presented. The maximal ideals in the Banach space of bounded real valued $^{-}$Quasicontinuous functions defined on $[0,1]$ are investigated.

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Introduction: In this paper, it is shown that the set of all bounded real $^{-}$Quasicontinuous functions defined on $[0,1]$ forms a commutative Banach algebra with identity under the supremum norm. The maximal ideals in this Banach algebra are identified to be of the form $M_x = \{f \,/\, f(x) = 0\}$ or $M_x^{-} = \{f \,/\, f(x-) = 0\}$ for $x \in [0,1]$.

In what follows, I and J stand for the real line, the unit closed interval $[0,1]$ and any closed and bounded interval $[a,b]$ respectively.

1. Preliminaries

1.1 Definition: Let $f : J \to \mathbb{R}$. We define $f(a-) = f(a)$ and $f(b+) = f(b)$. We say that $f(p+)$ exists at $p \in [a,b)$ and we write $f(p+) = L$, where $L \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon \ \forall \ x \in (p, p+\delta) \subset J$ Similarly for $p \in (a,b]$ we write $f(p-) = l \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon \ \forall \, x \in (p-\delta, p) \subset J$

1.2 Definition: A function $f : J \to \mathbb{R}$ is said to be $^{-}$Quasicontinuous on J if

(i) $f(p-)$ exists at every $p \in (a,b]$

(ii) $f(a+) = f(a)$

1.3 Definition: A function $f : J \to \mathbb{R}$ is said to be cliquish at a point $p \in J$ if for every $\varepsilon > 0$ and every neighborhood U of p in J there exists a non-empty open set $W \subset U$ such that

$\left| f(x) - f(y) \right| < \varepsilon \quad \forall\, x, y \in W$. We say that f is cliquish on J if it is cliquish at every point of J.

1.4 Definition: A mapping T from a linear space $\mathscr{V}$ into a linear space $\mathscr{W}$ is said to be linear if $T(cx + dy) = cT(x) + dT(y)$ for all x and y in $\mathscr{V}$ and constants c and d.

1.5 Definition: Let $\mathscr{V}$ and $\mathscr{W}$ be normed linear spaces. A linear map $T : \mathscr{V} \to \mathscr{W}$ is said to be bounded if there exists a real number $K \geq 0$

such that $\|T(x)\| \leq K\|x\| \quad \forall\ x \in \mathscr{V}$.

1.6 Definition: A linear functional on a vector space $\mathscr{V}$ over a field $\mathscr{K}$ is a linear mapping from $\mathscr{V}$ to $\mathscr{K}$.

2. Properties of $^{-}$Quasicontinuous functions

2.1 Proposition: Let $c \in \mathbb{R}$. If $f : J \to \mathbb{R}$ and $g : J \to \mathbb{R}$ are $^{-}$Quasicontinuous on J then $f+g$, cf, fg, $f \vee g$ and $f \wedge g$ are $^{-}$Quasicontinuous on J, where $(f \vee g)(x) = \max\left\{ f(x), g(x) \right\}$ and $(f \wedge g)(x) = \min\left\{ f(x), g(x) \right\}$.

Proof: Let $p \in (a,b]$. (i) Let $\varepsilon > 0$ be given. Then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that

$\left| f(x) - f(p-) \right| < \dfrac{\varepsilon}{2} \ \ \forall\, x \in (p - \delta_1, p) \subset J$ and $\left| g(x) - g(p-) \right| < \dfrac{\varepsilon}{2} \ \ \forall\, x \in (p - \delta_2, p) \subset J$.

Put $\delta = \min\{\delta_1, \delta_2\}$.

$$\text{Then } x \in (p-\delta, p) \Rightarrow \left|(f+g)(x) - (f(p-) + g(p-))\right| \leq \left| f(x) - f(p-) \right| + \left| g(x) - g(p-) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|(f+g)(x) - (f(p-) + g(p-))\right| < \varepsilon \quad \forall\, x \in (p-\delta, p)$$

Hence $(f+g)(p-)$ exists and $(f+g)(p-) = f(p-) + g(p-)$. Since f and g are continuous at a, $f+g$ is continuous at a.

Hence $f+g$ is $^{-}$Quasicontinuous on J.

(ii) If $c = 0$ then $cf = O$, where $O : J \to \mathbb{R}$ is defined by $O(x) = 0$.

Then cf is $^{-}$Quasicontinuous on J. Now suppose that $c \neq 0$.

Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that

$$|f(x) - f(p-)| < \frac{\varepsilon}{|c|} \;\; \forall\, x \in (p-\delta, p) \subset J$$

$$\Rightarrow \; |(cf)(x) - (cf)(p-)| < \varepsilon \;\; \forall \;\; x \in (p-\delta, p)$$

Hence $(cf)(p-)$ exists and $(cf)(p-) = c\, f\,(p-)$. Since f is continuous at a, cf is continuous at a.

Hence cf is $^{-}$Quasicontinuous on J.

(iii) Since f and g are $^{-}$Quasicontinuous at p, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|f(x) - f(p-)| < \varepsilon$ and $|g(x) - g(p-)| < \varepsilon \;\; \forall\, x \in (p-\delta, p) \subset J$

$$\begin{aligned}
\Rightarrow |(fg)(x) - f(p-)g(p-)| &= |f(x)g(x) - f(x)g(p-) + f(x)g(p-) - f(p-)g(p-)| \\
&\leq |f(x)||g(x) - g(p-)| + |g(p-)||f(x) - f(p-)| \\
&< |f(x)|\varepsilon + |g(p-)|\varepsilon \;\; \forall\, x \in (p-\delta, p) \\
&= |f(x) - f(p-) + f(p-)|\varepsilon + |g(p-)|\varepsilon \\
&< \varepsilon\left(\varepsilon + |f(p-)| + |g(p-)|\right) \;\; \forall\, x \in (p-\delta, p).
\end{aligned}$$

Hence $(fg)(p-)$ exists and $(fg)(p-) = f(p-)\, g(p-)$. Since f and g are continuous at a, fg is continuous at a.

Hence fg is $^{-}$Quasicontinuous on J.

It is easy to verify that $f \vee g$ and $f \wedge g$ are $^{-}$Quasicontinuous on J and we have the following.

$(f \vee g)(p-) = \max\{f(p-), g(p-)\}$ and $(f \wedge g)(p-) = \min\{f(p-), g(p-)\}$.

2.2 Proposition: Let $f_n : J \to \mathbb{R}$, $n = 1, 2, 3, \ldots,$ be $^{-}$Quasicontinuous on J and $f_n \to f$ uniformly on J. Then f is $^{-}$Quasicontinuous on J.

Proof: Let $p \in (a, b]$. Let $\varepsilon > 0$ be given. Then there exists an integer N such that $n \geq N$

$$\Rightarrow \; |f_n(x) - f(x)| < \frac{\varepsilon}{2} \quad \forall \;\; x \in J.$$

Since f_N is $^{-}$Quasicontinuous at p, there exists a $\delta > 0$ such that

$$|f_N(x) - f_N(p-)| < \varepsilon \;\; \forall\, x \in (p-\delta, p) \subset J$$

$$\begin{aligned}
x \in (p-\delta, p) \Rightarrow \; |f(x) - f_N(p-)| &= |f(x) - f_N(x) + f_N(x) - f_N(p-)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - f_N(p-)|
\end{aligned}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f_N(p-)| < \varepsilon \ \forall x \in (p-\delta, p) \subset J.$$

Hence $f(p-)$ exists for every $p \in (a,b]$.

Since each f_n is continuous at a and $f_n \to f$ uniformly on J, f is continuous at a. Hence f is $^{-}$Quasicontinuous on J.

2.3 Remark: It is not necessary that a $^{-}$Quasicontinuous function defined on a compact domain is bounded. It can be seen from the following example.

2.4 Example: Define $f : [-1,1] \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & if \quad -1 \le x \le 0 \\ \frac{1}{x} & if \quad 0 < x \le 1 \end{cases}$

This function f is $^{-}$Quasicontinuous on $[-1,1]$ but it is not bounded.

2.5 Remark: We denote the set of all bounded real valued $^{-}$Quasicontinuous functions defined on I by the symbol $\mathcal{QC}^{-}(I)$. By the propositions 2.1 and 2.2 it follows that $\mathcal{QC}^{-}(I)$ forms a commutative Banach algebra with identity under the supremum norm, where the identity $\boldsymbol{e} : I \to \mathbb{R}$ is defined by $\boldsymbol{e}(x) = 1 \ \forall \ x \in I$.

2.6 Proposition: Let $f : J \to \mathbb{R}$ and $p \in J$. If $f(p-)$ exists then f is cliquish at p.

Proof: Let $\varepsilon > 0$ be given and let U be a neighborhood of p in J. Then there exists a $\delta_1 > 0$ such that $(p-\delta_1, p+\delta_1) \bigcap J \subset U$.

Given $f(p-)$ exists. So there exists a $\delta_2 > 0$ such that

$$|f(x) - f(p-)| < \frac{\varepsilon}{2} \quad \forall \ x \in (p-\delta_2, p) \subset J.$$

Put $\delta = \min\{\delta_1, \delta_2\}$ and $W = (p-\delta, p)$.

Then for $x, y \in W$, we have $|f(x) - f(y)| = |f(x) - f(p-) + f(p-) - f(y)|$

$$\le |f(x) - f(p-)| + |f(y) - f(p-)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Thus for every $\varepsilon > 0$ and every neighborhood U of p, there exists a non-empty open set $W \subset U$ such that $\left| f(x) - f(y) \right| < \varepsilon \quad \forall \ x, y \in W$

$\Rightarrow f$ is cliquish at p.

2.7 Remark: From the above proposition it is clear that every $^{-}$Quasicontinuous function is cliquish. The converse is not true as is evident from the following example.

2.8 Example: Define f: $[-1,1] \to \mathbb{R}$ as follows.

$$\mathsf{f}(x) = \begin{cases} \dfrac{1}{x} & if \quad -1 \le x < 0 \\ 0 & if \quad 0 \le x \le 1 \end{cases}$$

Clearly f is cliquish at $x = 0$ but it is not $^{-}$Quasicontinuous .

2.9 Theorem [2]: If $f : J \to \mathbb{R}$ is $^{-}$Quasicontinuous then the set of points of discontinuity of f is atmost countable.

3. Maximal Ideals in $\mathcal{QC}^{-}(I)$

3.1 Definition: For each $x \in I$, we define the following.

(a) $M_x = \left\{ f \in \mathcal{QC}^{-}(I) \,/\, f(x) = 0 \right\}$ (b) $M_x^{-} = \left\{ f \in \mathcal{QC}^{-}(I) \,/\, f(x-) = 0 \right\}$.

3.2 Proposition: For each $x \in I$, the sets M_x and M_x^{-} are maximal ideals in the commutative Banach algebra $\mathcal{QC}^{-}(I)$.

Proof: For $x \in I$, define F_x and F_x^{-} on $\mathcal{QC}^{-}(I)$ by $F_x(f) = f(x)$ and $F_x^{-}(f) = f(x-)$ for $f \in \mathcal{QC}^{-}(I)$.

Clearly F_x and F_x^{-} are multiplicative linear functionals in the dual space $\mathcal{B}_{-}$ with kernels M_x and M_x^{-} respectively. Hence M_x and M_x^{-} are ideals. Moreover M_x and M_x^{-} are maximal ideals in $\mathcal{QC}^{-}(I)$.

3.3 Proposition: If M is a maximal ideal in $\mathcal{QC}^{-}(I)$ then either $M = M_x$ or $M = M_x^{-}$ for some $x \in I$.

Proof: For $x \in I$, define F_x and F_x^{-} on $\mathcal{QC}^{-}(I)$ by $F_x(f) = f(x)$ and $F_x^{-}(f) = f(x-)$ for $f \in \mathcal{QC}^{-}(I)$.

Clearly F_x and F_x^{-} are multiplicative linear functionals in the dual space $\mathcal{B}_{-}$ with kernels M_x and M_x^{-} respectively. Hence M_x and M_x^{-} are ideals. Moreover M_x and M_x^{-} are maximal ideals in $\mathcal{QC}^{-}(I)$.

3.3 Proposition: If M is a maximal ideal in $\mathcal{QC}^{-}(I)$ then either $M = M_x$ or $M = M_x^{-}$ for some $x \in I$.

Proof: Assume that $M \ne M_x$ and $M \ne M_x^{-}$ for any $x \in I$.

Then there exist f_x and g_x in M such that $f_x \notin M_x$ and $g_x \notin M_x^-$.

Define $\varphi_x : I \to \mathbb{R}$ by $\varphi_x(t) = f_x^2(t) + g_x^2(t-) \quad \forall \; t \in I$.

Clearly $\varphi_x \in \mathscr{QC}^-(I)$. Since φ_x is $^-$Quasicontinuous at x and $\varphi_x(x) > 0$, there exists a $\delta_x > 0$ such that $\varphi_x(t) > 0 \quad \forall \; t \in (\delta_x, 1]$ and for $x \neq 0$.

We have $\varphi_0(t) = f_0^2(t) + g_0^2(t-) \;\; \forall \, t \in I$.

Since φ_0 is continuous at 0 and $\varphi_0(0) > 0$ there exists a $\delta > 0$ such that $\varphi_0(t) > 0 \;\; \forall \; t \in [0, \delta)$. Then $[0,1] = \left(\bigcup_{x \neq 0} (\delta_x, 1] \right) \bigcup [0, \delta)$. Since I is compact, there exists an $x_0 \neq 0$ in I such that $[0,1] = (\delta_{x_0}, 1] \bigcup [0, \delta)$.

Put $\varphi = \varphi_{x_0}^2 + \varphi_0^2$. Then $\varphi \in M$ and $\varphi(t) > 0 \; \forall \; t \in I \;\; \Rightarrow \dfrac{1}{\varphi} \in M$.

Then $\boldsymbol{e} = \varphi . \dfrac{1}{\varphi} \in M$. This is a contradiction, Hence it follows that $M = M_x$ or $M = M_x^-$ for some $x \in I$.

3.4 Remark: Let $\mathscr{M}_-$ be the space of all maximal ideals in $\mathscr{QC}^-(I)$. Then $\mathscr{M}_-$ is a compact Hausdorff space with the weak* topology on $\mathscr{QC}^-(I)$. Hence $\mathscr{M}_-^2 = \mathscr{M}_- \times \mathscr{M}_-$ is a compact Hausdorff space with the product topology on $\mathscr{QC}^-(I) \times \mathscr{QC}^-(I)$.

3.5 Proposition: Let $\mathscr{A}^- = \left\{ \left(M_x, M_x^- \right) / \, x \in I \right\}$. Then there exists a one-to-one correspondence between I and $\mathscr{A}^-$.

Proof: Define $\Psi^- : I \to \mathscr{A}^-$ by $\Psi^-(x) = (M_x, M_x^-)$.

Clearly Ψ^- is surjective. If $0 \leq s < t \leq 1$, the function

$$\Psi_0^-(p) = \begin{cases} 0 & if \quad 0 \leq p \leq t \\ \dfrac{1}{p-t} & if \quad t < p \leq 1 \end{cases}$$

satisfies $\Psi_0^- \in M_t$ and $\Psi_0^- \notin M_s$.

$\Rightarrow \quad M_s \neq M_t$

$\Rightarrow \quad (M_s, M_s^-) \neq (M_t, M_t^-)$

$\Rightarrow \quad \Psi^-(s) \neq \Psi^-(t)$

Hence Ψ^- is $1-1$.

Hence Ψ^- is a one-to-one correspondence between I and $\mathscr{A}^-$.

3.6 Remark: Each maximal ideal in $\mathscr{OC}^{-}(I)$ is the kernel of some multiplicative linear functional on $\mathscr{OC}^{-}(I)$, hence can be identified with a multiplicative linear functional on $\mathscr{OC}^{-}(I)$. Let M_x and M_x^{-} be identified with the multiplicative linear functional F_x and F_x^{-} respectively. So we can write

$\mathscr{A}^{-} = \left\{ \left(F_x, F_x^{-} \right) / \, x \in I \right\}$.

3.7 Proposition: $\mathscr{A}^{-}$ is closed in $\mathcal{B}_{-}^{2} = \mathcal{B}_{-} \times \mathcal{B}_{-}$ and hence compact.

Proof: We prove that $\mathscr{A}^{-}$ is closed. Compactness is an immediate consequence of the Banach – Alaoglu theorem [5]. If $F = (F_1, F_2) \in \mathcal{B}_{-}^{2}$, we define $\|F\| = \max\left\{ \|F_1\|, \|F_2\| \right\}$. Then $\mathcal{B}_{-}^{2}$ is a Banach space under the above norm.

Let $S = \left\{ F / \|F\| \le 1 \right\} \subseteq \mathcal{B}_{-}^{2}$. Put $\mathcal{A} = \mathscr{A}^{-} \bigcup \{O\}$.

Then $\mathscr{A}^{-} \subset \mathscr{M}_{-}^{2} \subset \mathcal{A} \subset S \subset \mathcal{B}_{-}^{2}$.

Define $\mathcal{P}^{-} : \mathcal{A} \to \mathbb{R}$ by

$$\mathcal{P}^{-}(F) = \begin{cases} 1 & if \quad F \in \mathcal{A} \quad and \quad F \neq O \\ 0 & if \quad F = O \end{cases}$$

Since $\mathcal{P}^{-}$ is continuous, $\mathscr{A}^{-}$ and $\mathcal{A}$ are closed in $\mathcal{B}_{-}^{2}$.

4. Further Properties

4.1 Proposition: Fix $f \in \mathscr{OC}^{-}(I)$. Define $\psi_f : I \to \mathbb{R}^2$ by $\psi_f(x) = (f(x), f(x-))$, where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is considered with the norm $\|(x_1, x_2)\| = \max\left\{ |x_1|, |x_2| \right\}$. Then ψ_f is continuous on I if and only if f is continuous on I.

Proof: Assume that ψ_f is continuous on I. Let $p \in I$ and let $\varepsilon > 0$ be given. Since ψ_f is continuous at p, there exists a $\delta > 0$ such that $\left\| \psi_f(x) - \psi_f(p) \right\| < \varepsilon \;\; \forall \; x \in (p-\delta, p+\delta) \bigcap I$.

$\Rightarrow \; \|(f(x), f(x-)) - (f(p), f(p-))\| < \varepsilon \quad \forall \;\; x \in (p-\delta, p+\delta) \bigcap I$

$\Rightarrow \|(f(x) - f(p), f(x-) - f(p-))\| < \varepsilon \quad \forall \;\; x \in (p-\delta, p+\delta) \bigcap I$

$\Rightarrow \; \max\left\{ |f(x) - f(p)|, |f(x-) - f(p-)| \right\} < \varepsilon \quad \forall \;\; x \in (p-\delta, p+\delta) \bigcap I$

$\Rightarrow |f(x) - f(p)| < \varepsilon \quad \forall \;\; x \in (p-\delta, p+\delta)$

$\Rightarrow \quad f$ is continuous at p.

Thus if ψ_f is continuous at p then f is continuous at p.

Conversely suppose that f is continuous on I.

Then $\psi_f(x) = (f(x), f(x))\ \forall\ x \in I$.

Hence ψ_f continuous on I.

4.2 Proposition: Let $\mathbf{B} = \{\psi_f \,/\, f \in \mathcal{OC}^-(I)\}$. Define $F : \mathcal{OC}^-(I) \to \mathbf{B}$ by $F(f) = \psi_f$. Then F is a one-to-one continuous multiplicative linear mapping from $\mathcal{OC}^-(I)$ onto **B**.

Proof: Clearly $F : \mathcal{OC}^-(I) \to \mathbf{B}$ is surjective.

For $f, g \in \mathcal{OC}^-(I)$, $\psi_{f+g}(x) = ((f+g)(x), (f+g)(x-))$

$$= (f(x), f(x-)) + (g(x), g(x-))$$

$$= \psi_f(x) + \psi_g(x) \quad \forall\ x \in I$$

Hence $\psi_{f+g} = \psi_f + \psi_g \quad \forall \quad f, g \in \mathcal{OC}^-(I)$

$\Rightarrow \quad F(f+g) = F(f) + F(g)\ \forall \quad f, g \in \mathcal{OC}^-(I)$.

Let $c \in \square$. It is easy to see that $F(cf) = \psi_{cf} = c\psi_f = cF(f) \quad \forall \quad f \in \mathcal{OC}^-(I)$.

Hence F is linear.

Also we have $\psi_{fg}(x) = ((fg)(x), (fg)(x-))$

$$= (f(x), f(x-))\ (g(x), g(x-))$$

$$= \psi_f(x)\,\psi_g(x) \quad \forall \quad x \in I.$$

Hence $F(fg) = \psi_{fg} = \psi_f \psi_g = F(f)F(g)$.

$\Rightarrow F$ is multiplicative. Now we prove that F is $1-1$. For this, suppose that

$$F(f) = F(g) \;\Rightarrow\; \psi_f = \psi_g$$

$$\Rightarrow \psi_f(x) = \psi_g(x) \quad \forall \quad x \in I$$

$$\Rightarrow (f(x), f(x-)) = (g(x), g(x-))\ \forall\ x \in I$$

$$\Rightarrow f(x) = g(x) \quad \forall \quad x \in I$$

$$\Rightarrow f = g.$$

Hence F is $1-1$.

Suppose that $f_n \in \mathcal{OC}^-(I)$, $n = 1, 2, 3, \ldots$, and $f \in \mathcal{OC}^-(I)$.

Let $f_n \to f$ uniformly on I. Then for a given $\varepsilon > 0$ there exists an integer $N > 0$ such that

$|f_n(x) - f(x)| < \dfrac{\varepsilon}{3}$ for all $n \geq N$ and all $x \in I$.

Fix $x \in I$ and $n \geq N$. Since f_n is $^{-}$Quasicontinuous there exists a $\delta_1 > 0$ such that

$$|f_n(t) - f_n(x-)| < \frac{\varepsilon}{3} \;\; \forall \;\; t \in (x - \delta_1, x).$$

Since f is also $^{-}$Quasicontinuous at x, there exists a $\delta_2 > 0$ such that

$$|f(t) - f(x-)| < \frac{\varepsilon}{3} \;\; \forall \;\; t \in (x - \delta_2, x).$$

Put $\delta = \min\{\delta_1, \delta_2\}$. Then for $t \in (x - \delta, x)$ and $n \geq N$,

$$\begin{aligned} |f_n(x-) - f(x-)| &= |f_n(x-) - f_n(t) + f_n(t) - f(t) + f(t) - f(x-)| \\ &\leq |f_n(x-) - f_n(t)| + |f_n(t) - f(t)| + |f(t) - f(x-)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $|f_n(x-) - f(x-)| < \varepsilon$ for all $n \geq N$ and all $x \in I$.

$$\begin{aligned} n \geq N \;\; \Rightarrow \;\; \|F(f_n) - F(f)\| &= \|\psi_{f_n} - \psi_f\| \\ &= \sup\left\{\|\psi_{f_n}(x) - \psi_f(x)\| / x \in I\right\} < \varepsilon. \end{aligned}$$

$\Rightarrow$ $F(f_n) \to F(f)$ Uniformly on I.

Hence F is continuous on $\mathscr{QC}^{-}(I)$.

4.3 Proposition: The set $\mathbf{B} = \{\psi_f \,/\, f \in \mathscr{QC}^{-}(I)\}$ is a commutative Banach algebra with identity ψ_e under the norm defined by $\|\psi_f\| = \sup\left\{\|\psi_f(x)\| / x \in I\right\}$, where $\psi_e(x) = (1,1) \;\; \forall \;\; x \in I$.

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