Volume 1, No. 3, March 2013
Journal of Global Research in Mathematical Archives
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

ON ASYMMETRIC METRIC SPACES

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Abstract: In this literature, we study asymmetric metric spaces. Next, we prove some important and practical results in these spaces. Finally, we obtain an interesting result by the concept of denseness in asymmetric metric spaces

.**Keywords**: Asymmetric metric spaces, Forward and backward limits, Forward and backward denseness.

MSC2010: 26A15

1. INTRODUCTION

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric *d* has to satisfy $d(x, y) = d(y, x)$.There are many applications of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton–Jacobi equations [1] in mind.

The study of asymmetric metrics apparently goes back to Wilson [2]. Following his terminology, asymmetric metrics are often called quasi-metrics. Author in [3] has discussed on asymmetric metric spaces. Also, Aminpour, Khorshidvandpour and Mousavi [4] have proved interesting theorems.

In this paper, we prove some theorems in asymmetric metric spaces. We start with some definitions from [3].Also we extend some results in [5].

Definition 1.1. A function $d : X \times X \to \mathbb{R}$ is an *asymmetric metric* and (X, d) is an *asymmetric metric space* if:

(1) For every $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0$ holds if and only if $x = y$,

(2) For every $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

Henceforth, (X, d) shall be an asymmetric metric space.

Example1.2. Consider $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by

$$d(x, y) = \begin{cases} x - y & x \geq y \\ \alpha(y - x) & y > x \end{cases}$$

Then d is an asymmetric metric on $\mathbb{R}$.

Definition1.3. The *forward topology* $\tau+$ induced by *d* is the topology generated by the *forward open balls*

$$B^+(x, \varepsilon) = \{y \in X: d(x, y) < \varepsilon\} \quad \text{for } x \in X,\ \varepsilon > 0.$$

Likewise, the *backward topology* $\tau-$ induced by *d* is the topology generated by the *backward open balls*

$$B^{-}(x,\varepsilon) = \{y \in X: d(y,x)<\varepsilon\} \quad \text{for } x \in X,\ \varepsilon > 0.$$

Definition 1.4. A sequence $\{x_k\}_{k\in\mathbb{N}}$ *forward converges* to $x_0 \in X$, respectively *backward converges* to $x_0 \in X$ if and only if

$$\lim_{k\to\infty} d(x_0, x_k) = 0, \quad \text{Respectively} \quad \lim_{k\to\infty} d(x_k, x_0) = 0.$$

Then we write $x_k \xrightarrow{f} x_0$, $x_k \xrightarrow{b} x_0$ respectively.

Example1.5. Let $(\mathbb{R}, d)$ be an asymmetric space, where d is as in example1.2. It is easy to show that the sequence $\{x + 1n\ n\in\mathbb{N}$ $(x\in X)$ is both forward and backward converges to x.

Definition 1.6 Suppose (X,d_X) and (Y, d_Y) are asymmetric metric spaces. Let $f: X \to Y$ *be a function* . We say f is *forward continuous at* $x\in X$, respectively *backward continuous*, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y\in B^{+}(x,\delta)$ implies $f(y) \in B^{+}(f(x),\varepsilon)$, respectively $f(y) \in B^{-}(f(x),\varepsilon)$.
However, note that uniform forward continuity and uniform backward continuity are the same.

Lemma1.8. *Let* $d: X \times X \to \mathbb{R}^{\geq 0}$ *be an asymmetric metric. If (X, d) is forward sequentially compact and* $x_n \xrightarrow{b} x_0$ *then* $x_k \xrightarrow{f} x_0$.

Notation1.9. We introduce some further notations. Y^X denotes the space of functions from X to Y. The *uniform metric* on Y^X is

$$\bar{\rho}(f,g) := \sup\{\bar{d}(f(x), g(x)) : x \in X\},$$

where $\bar{d}(x,y) := \min\{d(x,y), 1\}$ and d is the asymmetric metric associated with Y.

2.MAIN RESULTS

Theorem 2.1. Let (X,d)be an asymmetric metric space . Then $x_n \xrightarrow{f} x$ if only if each subsequence of it be forward convergent to x .

Proof. Let $x_n \xrightarrow{f} x$ .Given $\varepsilon > 0$, there exists$N \in N$ such that $d(x, x_n) < \varepsilon$ for all $n \geq N$. Suppose that$\{x_{n_k}\}_{k=1}^{\infty}$ be an arbitrary subsequence of $\{x_n\}_{n=1}^{\infty}$. If $n_k \geq N$we have$d(x, x_{n_k}) < \varepsilon$, i . e ,$x_{n_k} \xrightarrow{f} x$ .Conversely .since$\{x_n\}$ is a subsequence of itself ,so $x_{n_k} \xrightarrow{f} x$. □

Remark 2.2. One can rewrite the previous theorem for back limits .

Theorem2.3. Let (X,d) be an asymmetric metric space . If X is backward sequentially compact and $x_n \xrightarrow{f} x$, then $x_n \xrightarrow{b} x$.

Proof. Let $x_n \xrightarrow{f} x$ .Since $X$is backward sequentially compact so by theorem 2.1 each subsequence of $\{x_n\}_{n\in N}$, namely $\{x_{n_k}\}$, is backward convergent to x .On the other hand , $\{x_{n_k}\}_{k\in N}$, has a subsequence which backward convergent , say $\{x_{n_{k_j}}\}_{j\in N}$. So $x_{n_{k_j}} \xrightarrow{b} y$ .Now by [1,lemma 3.1] , we deduce that $x = y$. We show that $x_n \xrightarrow{b} x$. Let $x_n \nrightarrow x$ .Then there exists a $\varepsilon_0 > 0$ a subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$so that $d(x_{n_k}, x) \geq \varepsilon_0$ for each $K \in N$. Also , $\{x_{n_k}\}_{k\in N}$, itself , has a subsequence which is backward convergent to x , say $\{x_{n_{k_j}}\}_{j\in N}$ hence we can find $J \in N$ such that $d(x_{n_{k_j}}, x) < \varepsilon_0$ for $j \geq J$ which is a contradiction .So $x_n \xrightarrow{b} x$. □

Lemma 2.4. If backward convergence implies the forward convergence of a sequence , then the backward limit is unique .

Proof. Let $x_n \overset{b}{\to} x$implies $x_n \overset{f}{\to} y \in X$. Also , suppose that$x_n \overset{b}{\to} z$. Given > 0 , there exists$N_1 \in N$so that $d(y, x_k) < \frac{\varepsilon}{2}$ for all $K \geq N_1$. On the other hand , there exists $N_2 \in N$ such that $d(x_k, x) < \frac{\varepsilon}{2}$ for all $K \geq N_2$ by lemma [1,lemma 3.1] , we deduce that $y = z$. Set $N := max\{N_1, N_2\}$then we have

$$d(z,x) \leq d(z,x_k) + d(x_k,x) < \varepsilon$$

Since $\varepsilon > 0$was arbitrary , so $z = x$. □

Remark 2.5. Auther in [1] has proved a similar lemma by replacing forward by backward .

Theorem 2.6. Let (X, d) be a backward totally bounded asymmetric metric space which the backward convergence of a sequence implies the forward convergence . Then X is sequentially compact .

Proof. Suppose that$\{x_n\}_{n \in N}$ be an arbitrary sequence in . Given $\varepsilon > 0$,there exist$y_1, y_2, \dots \dots, y_k$ in X such that

$$X = \bigcup_{i=1}^{k} B^-(y_i, \varepsilon)$$

Also , we can find $N \in \mathbb{N}$ and $1 \leq j \leq k$ so that $\{x_n\} \subset B^-(y_j, \varepsilon)$ for all $n \geq \mathbb{N}$. Hence $x_n \overset{b}{\to} y_j$ .It is easy to show that,y_j is unique. Now, by assumption we have

$$x_n \overset{f}{\to} y_j$$

Since $\{x_n\}$ is a subsequence of itself , then (X, d) is forward and backward sequentially compact, as desired□

Note2.7. I has introduced the concept of denseness in [6]. In the case that X is both of forward and backward compact and it has a forward and backward dense subset, then all of results in the work come back to metric space.

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