Volume 4, No.12, December 2017

Journal of Global Research in Mathematical Archives

UGC Approved Journal

RESEARCH PAPER

Available online at http://www.jgrma.info

ISSN 2320 – 5822

NEWTON'S LAW OF COOLING -LAPLACE TRANSFORM

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Abstract: In this paper, I solved the problems occurred in first order ordinary linear differential equations based on Newton's Law of cooling by using Laplace Transform.

Keywords: Laplace transform, Newton's Law of Cooling, Differential Equations

I. INTRODUCTION

The theory of Laplace Transform is an important part of the mathematical surroundings required by engineers, physicists and mathematicians. It gives an easy and successful means for solving certain types of differential and integral equations.
The Laplace transform reduces the problem of solving differential equations to an algebraic problem. It is particularly useful for solving problems where the mechanical or electrical driving force has discontinuities, An integral transform called the Laplace transform defined for function of exponential order, we consider in the set A defined by

$$A=\left\{\{f(t):\exists M,\mu_1,\mu_2>0,|f(t)|<Me^{\frac{|t|}{\mu_j}},if\ t\in(-1)^j\times[0,\infty)\right\}$$, Given function in the set M must be finite number, μ_1,μ_2 may be finite (or) infinite. The Laplace Transformations (L.T.) is a special case of an integral transformation and is defined by

$$f(s)=L[f(t)]=\int_0^\infty e^{-st}f(t)dt \_\_\_\_\_\_\_\_\_\_(1)$$

The satisfactory condition for the survival of the Laplace transform are that f(t) for $t\geq 0$ be piecewise continuous and of the exponential order otherwise Laplace transform may (or) may not exist.
The unique function $f(t)=L^{-1}\{f(s)\}$ is called inverse Laplace transform of $\overline{f}(s)$_____(2)

II. Laplace Transform of standard functions and properties

(a) $L(e^{-at})=\dfrac{1}{s+a}$ **(s>a)**

Proof: we have $L(e^{-at})=\int_0^\infty e^{-st}e^{-at}dt=\int_0^\infty e^{-(s+a)t}dt=\dfrac{-1}{(s+a)}[e^{-(s+a)t}]_0^\infty=\dfrac{1}{s+a}$

If a=0 the $L(1)=\frac{1}{s}$ (s>0)

Also changing the sign of ‘a’ we get $L(e^{at})=\frac{1}{s-a}$

(b) $L(\sin at)=\frac{a}{s^2+a^2}$ **and** $L(\cos at)=\frac{s}{s^2+a^2}$ **(s>0)**

Proof: $L(\cos at+i\sin at)=L(e^{iat})=\frac{1}{s-ia}=\frac{s+ia}{s^2+a^2}$

Equating real and imaginary parts we get

$$L(\sin at)=\frac{a}{s^2+a^2}, L(\cos at)=\frac{s}{s^2+a^2}$$

(c) $L(\sinh at)=\frac{a}{s^2-a^2}$ **and** $L(\cosh at)=\frac{s}{s^2-a^2}$ $(s>|a|)$

Proof:

$$L(\sinh at)=L\left(\frac{e^{at}-e^{-at}}{2}\right)=\frac{1}{2}L(e^{at})-\frac{1}{2}L(e^{-at})$$

$$=\frac{1}{2}[\frac{1}{s-a}-\frac{1}{s+a}], Thus \ \sin(hat)=\frac{a}{s^2-a^2}$$

Similarly, we can show that $L(\cosh at)=\frac{s}{s^2-a^2}$

(d) $L(t^n)=\frac{(n+1)}{s^{n+1}}[(n+1)>0 \, and \, s>0]$

Proof:

$$L(t^n)=\int_0^\infty e^{-st}t^n dt=\int_0^\infty e^{-p}\frac{p^n}{s^n}\frac{dp}{s} \qquad where\ p=st$$

$$=\int_0^\infty \frac{e^{-p}.p^n}{s^{n+1}}dp=\frac{\Gamma(n+1)}{s^{n+1}}$$

$$Since \ \ \Gamma(n+1)=\int_0^\infty e^{-x}x^n dx \quad and \quad \Gamma(n+1)=n! \ \ if \ \ n \ \ is \ \ a+ve\text{int}\,eger$$

$$\therefore L(t^n)=\frac{n!}{s^{n+1}}$$

from above we have

$$L(1)=\frac{1}{s}, \quad L(t)=\frac{1}{s^2}, \ L(t^2)=\frac{2}{s^3}, \quad L(t^3)=\frac{6}{s^4}$$

Linearity Property:

Let $f_1(t)$ and $f_2(t)$ be two functions defined on [0,∞) such that the Laplace transforms $L[f_1(t)]$ and $L[f_2(t)]$ exists. If k_1 and k_2 are two constants, then

$$L[k_1 f_1(t)+k_2 f_2(t)]=k_1L[f_1(t)]+k_2L[f_2(t)]$$

This property is valid since

$$\int_0^\infty \{k_1 f_1(t)+k_2 f_2(t)\}e^{-st}dt=\int_0^\infty k_1 f_1(t)e^{-st}dt+\int_0^\infty k_2 f_2(t)e^{-st}dt=k_1L[f_1(t)]+k_2L[f_2(t)]$$

This result can be extended to the linear combinations of more than two functions

Laplace transforms Derivatives:

Theorem: If f(t) is continuous $\forall t\geq 0$ and of exponential order, say σ and has a derivative f(t) which is piecewise continuous on every finite interval [0,N] for each N>0, then the Laplace Transform of the derivative f(t) exists for s>σ and

$$L\{f'(t)\} = sLf(t) - f(0)$$

Proof:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt \Rightarrow L[f'(t)] = \int_0^\infty e^{-st} f'(t)dt$$

Integration by parts gives $L[f'(t)] = \left[e^{-st} f(t)dt\right]_0^\infty + \int_0^\infty e^{-st} f(t)dt$

Since f(t) is of an exponential order, the integrand in first integral on the R.H.S. is zero at the upper limit when s>σ and if f(0) at the lower limit. Thus, we have

$$L[f'(t)] = 0 - f(0) + \int_0^\infty e^{-st} f(t)dt = sLf(t) - f(0)$$

III. NEWTON'S LAW OF COOLING

Newton's law of cooling states that the rate of change of temperature of a body is directly proportional to difference of the body temperature and its surrounding temperature.

Let θ be the temperature of body at any time t, θ_0 be its surrounding temperature. Then by Newton's law of cooling

$\frac{d\theta}{dt} \alpha (\theta - \theta_0) \Rightarrow -\frac{d\theta}{dt} = k(\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k\theta + k\theta_0$, where k is proportionality constant (k>0)

IV. APPLICATION OF LAPLACE TRANSFORM ON NEWTON'S LAW COOLING.

Example-1: The temperature of the body drops from 100^0c to 80^0c in 20 minutes when the surrounding temperature is at 20^0c. Then what will be the temperature after 30 minutes and when will be the temperature 45^0c.

Solution. Given that, the surrounding temperature $\theta_0 = 20^0 c$

Initial temperature at t=0, $\theta(0) = 100^0 c$ and at t=20 minutes, $\theta(20) = 80^0 c$

From Newton's Law of cooling, we have

$$\frac{d\theta}{dt} \alpha (\theta - \theta_0) \Rightarrow -\frac{d\theta}{dt} = k(\theta - \theta_0) \Rightarrow \frac{d\theta}{dt} = -k\theta + k\theta_0 \Rightarrow \frac{d\theta}{dt} + k\theta = 20k \quad \text{\_\_\_\_\_}(3)$$

From (1)

$$L\left\{\frac{d\theta}{dt} + k\theta(t)\right\} = L\{20k\}$$

$$\Rightarrow L\left\{\frac{d\theta}{dt}\right\} + kL\{\theta(t)\} = 20kL\{1\}$$

From Laplace Transform of derivatives

$$sL\{\theta(t)\} - \theta(0) + kL\{\theta(t)\} = 20kL\{1\}$$

$$(s+k)L\{\theta(t)\} = \frac{20k}{s} + 100$$

$$L\{\theta(t)\} = \frac{20k}{s(s+k)} + \frac{100}{(s+k)}$$

$$= \frac{20k}{k}\left[\frac{1}{s} - \frac{1}{s+k}\right] + \frac{100}{s+k}$$

$$= \frac{20}{s} - \frac{20}{s+k} + \frac{100}{s+k}$$

$$L\{\theta(t)\} = \frac{20}{s} + \frac{80}{s+k} -------(4)$$

From (2) we get,

$$\theta(t) = 20 + 80e^{-kt} --------(5)$$

We have at t=20 minutes, $\theta(20) = 80^0 c$, from (3)

$$80 = 20 + 80e^{-20k} \Rightarrow 60 = 80e^{-20k} \Rightarrow \frac{3}{4} = e^{-20k} \Rightarrow e^{-k} = \left(\frac{3}{4}\right)^{1/20} -----(6)$$

Now at t=30 minutes, from (5)

$$\theta(30) = 20 + 80e^{-30k} \Rightarrow 20 + 80(e^{-k})^{30}$$

From (6) , $\theta(30) = 20 + 80\left(\frac{3}{4}\right)^{3/2} = 71.96^0 c$

Now to find the time, to become $\theta(t) = 45^0 c$

From (5),

$$45 = 20 + 80e^{-kt} \Rightarrow 25 = 80(e^{-k})^t \Rightarrow \frac{25}{80} = \left(\frac{3}{4}\right)^{t/20}$$

Applying logarithm, we get

$$\frac{t}{20} = \ln\left(25/80\right) \Big/ \ln\left(3/4\right) \Rightarrow t \approx 80 \min utes$$

Example-2. (Estimation of time of murder)

The body of murder victim was discovered at 12:00PM the doctor took the temperature of body at 12:30Pm which was 94^{0}F. He took again temperature after one hour when it showed 93^{0}F, and noticed that the temperature of the room was 60^{0}F. Estimation the time of death (Normal temperature of human body=98.6^{0}F)

Solution. Given that, Room temperature $\theta_0 = 60^0 F$

The initial temperature of body is $94^0 F$

i.e., at t=0(12:30Pm), $\theta(0) = 94^0 F$

From Newton's law of cooling

We have $\frac{d\theta}{dt} + k\theta = k\theta_0$

$$\frac{d\theta}{dt} + k\theta(t) = 60k ------(7)$$

From(1), we get

$$L\left\{\frac{d\theta}{dt} + k\theta(t)\right\} = L\{60k\}$$

From Linear property

$$\Rightarrow L\left\{\frac{d\theta}{dt}\right\}+kL\{\theta(t)\}=60kL\{1\}$$

From Laplace Transform of derivatives

$$sL\{\theta(t)\}-\theta(0)+kL\{\theta(t)\}=\frac{60k}{s}$$

$$(s+k)L\{\theta(t)\}=\frac{60k}{s}+94$$

$$L\{\theta(t)\}=\frac{60k}{s(s+k)}+\frac{94}{(s+k)}$$

$$=60\left[\frac{1}{s}-\frac{1}{s+k}\right]+\frac{94}{s+k}$$

$$L\{\theta(t)\}=\frac{60}{s}+\frac{34}{s+k}--------(6)$$

From(2),we get

$$\theta(t)=60+34e^{-kt}--------(7)$$

Now at time t=60minutes (1:30Pm), $\theta(t)=93^0 F$, from (7)

$$93=60+34e^{-60k}\Rightarrow 33=34e^{-60k}\Rightarrow e^{-60k}=\left(\frac{33}{34}\right)\Rightarrow e^{-k}=\left(\frac{33}{34}\right)^{1/60}-----(8)$$

But the human body temperature before the death is 98.6^{0}F, from (7)

$$98.6=60+34e^{-kt}\Rightarrow 38.6=34(e^{-kt})\Rightarrow (e^{-kt})=\frac{38.6}{34}\Rightarrow (e^{-k})^t=\frac{38.6}{34}$$

From (8), $\left(\frac{33}{34}\right)^{t/60}=\left(\frac{38.6}{34}\right)$

Applying logarithm on both sides

$$\frac{t}{60}=\ln\left(38.6/34\right)\Big/\ln\left(33/34\right)\Rightarrow t=\frac{60\ln(\frac{38.6}{34})}{\ln\left(\frac{33}{34}\right)}=-255\min.$$

$$t\approx -4\,hours\,25\min.$$

therefore the estimated death time is 12:30-4:25=8:05 AM

Example-3.A cake is removed from an oven its temperature is measured at 300^{0}F, 3minutes later its temperature is 200^{0}F. How long it will take to cool off a room temperature is70^{0}F.

Solution. Given, room temperature $\theta_0=70^0 F$

At t=0, $\theta(0)=300^0 F$

From Newton's Law of cooling, we have

$$\Rightarrow \frac{d\theta}{dt}+k\theta=k\theta_0$$

$$\Rightarrow \frac{d\theta}{dt}+k\theta(t)=70k------(9)$$

Taking Laplace transform on both sides

$$L\left\{\frac{d\theta}{dt}+k\theta(t)\right\}=L\{70k\}$$

$$\Rightarrow L\left\{\frac{d\theta}{dt}\right\}+kL\{\theta(t)\}=70kL\{1\}$$

From Laplace Transform of derivatives

$$sL\{\theta(t)\}-\theta(0)+kL\{\theta(t)\}=\frac{70k}{s}$$

$$(s+k)L\{\theta(t)\}=\frac{70k}{s}+300$$

$$L\{\theta(t)\}=\frac{70k}{s(s+k)}+\frac{300}{(s+k)}----------(10)$$

Taking inverse Laplace Transform we get

$$\theta(t)=L^{-1}\left\{\frac{70k}{s(s+k)}\right\}+L^{-1}\left\{\frac{300}{(s+k)}\right\}$$

$$=L^{-1}\left\{\frac{70}{s}-\frac{70}{(s+k)}\right\}+L^{-1}\left\{\frac{300}{(s+k)}\right\}$$

$$=L^{-1}\left\{\frac{70}{s}+\frac{230}{(s+k)}\right\}$$

$$\theta(t)=70+230e^{-kt}--------(11)$$

Given at t=3minutes, $\theta(3)=200^0 F$, from(11)

$$200=70+230e^{-3k}\Rightarrow 130=230e^{-3k}\Rightarrow \frac{13}{23}=e^{-3k}\Rightarrow e^{-k}=\left(\frac{13}{23}\right)^{1/3}-----(12)$$

However the equation (11) given

No finite solution to $\theta(t)=70^0 F$

Since $\lim_{t\to\infty}\theta(t)=70$

We expect the cake will approaches to room temperature, which is given in the tabular form

$\theta(t)$ in ^0F	**Time(min)**
75	20.1
74	21.3
73	22.8
72	24.9
71	28.6
70.5	32.3

From above we can conclude that the cake will approximately be at room temperature in 30 minutes

V. CONCLUSION

We can apply Laplace transform to solve the problems related to Newton's Law of cooling.

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