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Approximate Controllability of Second-Order Neutral Evolution Equations with Nonlocal Conditions

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Abstract

In this paper, we consider a class of second-order neutral evolution equations with nonlocal conditions in Banach spaces. This paper deals with the approximate controllability for a class of second-order control systems. A set of sufficient conditions are established for the approximate controllability of a class of second-order neutral evolution equations with nonlocal conditions in Banach spaces and Schauder's fixed point theorem is used to prove the main result. An example is also given to illustrate the main result.

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1 Introduction

The concept of Controllability gains more attention in the past decade because of its various applications in the field of applied mathematics. Controllability generally means that with the help of set of admissible controls, it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state. Controllability can be distinguished as exact and approximate controllability. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be studied to an arbitrary small neighbourhood of final state, i.e., it gives the possibility of steering the system to state which forms the dense subspace in the state space. When compared with the exact one, the approximate controllaility is completely adequate in applications. Hence, it is necessary to concentrate more on this type of problems. For basic concepts about the controllability, reader may refer [1, 2, 5, 7, 8, 12, 16, 24–27, 32, 39, 41].

The second order differential equations play a vital role in constructing the various mathematical and physical model problems. There exists an extensive literature studies regarding the abstract second order problems. The concept of cosine family of functions is used to find the existence of solutions to the second order abstract cauchy problems in the case of autonomous problems. For basic concepts about the cosine function theory, we refer the reader to [14, 34–36, 40]. The existence of non-autonomous second order abstract cauchy problem corresponding to the family $\{A(t); t \in I\}$ is directly related to the concept of evolution operator generated by the family. The existence of second order evolution equations have been studied in various papers [4, 13, 17, 18, 21, 28, 33]. Since the introduction of the concept of nonlocal condition by Byszewski, nonlocal cauchy problems have been used in many mathematical models than the classical ones because of its better real world applications. Several authors studied the existence and controllability of differential equations with nonlocal conditions [4, 10, 11, 27].

Recently, in [25] Mahmudov et.al. studied the approximate controllability of second order evolution differential inclusions in Banach spaces by using Bohnenblust Karlin Fixed point theorem. In [4] Balachandran et al. discussed the nonlocal cauchy problem for second order integro-differential evolution equations in Banach spaces. In [15] Hernandez discussed the existence of solutions to a second order partial differential equation with nonlocal conditions. In [32], Sakthivel et al. studied the approximate controllability of second-order systems with state-dependent delay by using Schauder's fixed point theorem. Upto the authors knowkledge, there is no work reported on the approximate controllability of second-order evolution equation with nonlocal condition of the form (1.1)-(1.2). This is the main motivation of doing this work.

Inspired by the above works, in this paper, we establish sufficient conditions for the approximate controllability for a class of second-order neutral evolution differential equations with nonlocal conditions in Banach spaces of the form

$$\frac{d}{dt}[x'(t) - g\left(t, x(t), x'(t)\right)] = A(t)x(t) + f\left(t, x(t), x'(t)\right) + Bu(t), \quad t \in I = [0, b], \tag{1.1}$$

$$x(0) = x_0 + p(x, x'), \quad x'(0) = y_0 + q(x, x'), \tag{1.2}$$

In this equation, $A(t) : D(A(t)) \subseteq X \to X$ is a closed linear operator on a Banach space X with norm $\|\cdot\|$. Here U is a Banach space and B is a bounded linear operator from U to X and the functions. Also, $f, g : I \times X \times X \to X$, $p, q : \mathcal{C} \times \mathcal{C} \to X$ are the appropriate functions defined later.

We organize this paper as follows. In section 2, we give some necessary concepts and important definitions about the sine and cosine operator theory and evolution equations. In section 3, we establish the set of sufficient conditions for the approximate controllability for a class of second-order evolution differential equations with nonlocal conditions in Banach spaces. In section 4, we establish the set of sufficient conditions for the approximate controllability for a class of second-order evolution impulsive differential equations with nonlocal conditions in Banach spaces. An example is given 5 to illustrate the theory of the above found result.

2 Preliminaries

In this section, we mention a few results, notations and lemmas needed to establish our main results. We introduce certain notations which will be used throughout the article without any further mention. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, and $\mathcal{L}(Y, X)$ be the Banach space of bounded linear operators from Y into X equipped with its natural topology; in particular, we use the notation $\mathcal{L}(X)$ when $Y = X$. By $\rho(A)$, we denote the resolvent set of a linear operator A. Throughout this paper, $B_r(x, X)$ will denote the closed ball with center at x and radius $r > 0$ in a Banach space X. We denote by $\mathcal{C}$, the Banach space $C(J, X)$ endowed with supnorm given by $\|x\|_{\mathcal{C}} \equiv \sup_{t \in I} \|x(t)\|$, for $x \in \mathcal{C}$.

Now, we consider the abstract non-autonomous second order initial value problem

$$x''(t) = A(t)x(t) + f(t), \quad 0 \leq s, t \leq b, \tag{2.1}$$

$$x(s) = x_0, \quad x'(s) = y_0, \tag{2.2}$$

where $A(t) : D(A(t)) \subseteq X \to X$, $t \in I = [0, b]$ is a closed densely defined operator and $f : I \to X$ is an appropriate function. For detailed concepts about the evolution operator, the reader may refer [13, 21, 28, 29] and the references within. In the above mentioned works, the existence of solutions to the problem $(2.1) - (2.2)$ is related to the existence of an evolution operator $S(t, s)$ for the homogeneous equation

$$x''(t) = A(t)x(t), \quad 0 \le s, t \le b. \tag{2.3}$$

Here we assume that the domain of $A(t)$ is a subspace D dense in X and independent of t, and for each $x \in D$ the function $t \mapsto A(t)x$ is continuous. We will use the concept of evolution operator discussed by Kozak [19].

Definition 2.1. *A family S of bounded linear operators $S(t, s) : I \times I \to \mathcal{L}(X)$ is called an evolution operator for* (2.3) *if the following conditions are satisfied:*

(Z_1) *For each $x \in X$, the mapping $[0, b] \times [0, b] \ni (t, s) \to S(t, s)x \in X$ is of class C^1 and*

- (i) *for each $t \in [0, b]$, $S(t, t) = 0$,*
- (ii) *for all $t, s \in [0, b]$, and for each $x \in X$,*

$$\frac{\partial}{\partial t} S(t, s)x\Big|_{t=s} = x, \qquad \frac{\partial}{\partial t} S(t, s)x\Big|_{t=s} = -x.$$

(Z_2) *For all $t, s \in [0, b]$, if $x \in D(A)$, then $S(t, s)x \in D(A)$, the mapping $[0, b] \times [0, b] \ni (t, s) \to S(t, s)x \in X$ is of class C^2 and*

- (i) $\frac{\partial^2}{\partial t^2} S(t, x)x = A(t)S(t, s)x,$
- (ii) $\frac{\partial^2}{\partial s^2} S(t, x)x = S(t, s)A(s)x,$
- (iii) $\frac{\partial}{\partial s}\frac{\partial}{\partial t} S(t, x)x\Big|_{t=s} = 0.$

(Z_3) *For all $t, s \in [0, b]$, if $x \in D(A)$, then $\frac{\partial}{\partial s} S(t, s)x \in D(A)$, then $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t, s)x$, $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t, s)x$ and*

- (i) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(t, s)x = A(t)\frac{\partial}{\partial s} S(t, s)x,$
- (ii) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(t, s)x = \frac{\partial}{\partial t} S(t, s)A(s)x,$

and the mapping $[0, b] \times [0, b] \ni (t, s) \to A(t)\frac{\partial}{\partial s} S(t, s)x$ is continuous.

In the following work, we assume that there exists an evolution operator $S(t, s)$ associated to the operator $A(t)$. The following assumptions are made with the help of [4]. There exists a positive constants M, M^* and N, N^* such that

$$N = \sup\{\|S(t, s)\| : t, s \in I\}, \quad M = \sup\{\|C(t, 0) : t \in I\},$$

$$\text{and } N^* = \sup\left\{\|\frac{\partial}{\partial t} S(t, s)\| : t, s \in I\right\}, \quad M^* = \sup\left\{\|\frac{\partial}{\partial t} C(t, 0))\| : t \in I\right\}$$

respectively. Further, for $x \in X$ and $t_1, t_2, s \in I$,

$$\Big[\frac{\partial}{\partial t_1} C(t_1, 0) - \frac{\partial}{\partial t_2} C(t_2, 0)\Big]x \to 0, \quad \Big[\frac{\partial}{\partial t_1} S(t_1, s) - \frac{\partial}{\partial t_2} S(t_2, s)\Big]x \to 0, \text{as } t_1 \to t_2.$$

Given that $f : I \to X$ is an integrable function, the mild solution $x : [0, b] \to X$ of the problem $(2.1) - (2.2)$ is given by

$$x(t) = C(t, s)x_0 + S(t, s)y_0 + \int_0^t S(t, \tau)f(\tau)d\tau.$$

In the literature several techniques have been discussed to establish the existence of the evolution operator $S(\cdot, \cdot)$. In particular, a very studied situation is that $A(t)$ is the perturbation of an operator A that generates a cosine operator function and so we give some important properties of the theory of cosine functions. Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space X. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)x = \int_0^t C(s)xds, \quad x \in X, \quad t \in \mathbb{R}.$$

We refer the reader to [14, 34, 35] for the necessary concepts about cosine functions. It follows that

$$C(t)x - x = A \int_0^t S(s)xds,$$

for all X. The notation $[D(A)]$ stands for the domain of the operator A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. Moreover, in this paper the notation E stands for the space formed by the vectors $x \in X$ for which the function $C(\cdot)x$ is a class C^1 on $\mathbb{R}$. It was proved by Kisyński [18] that the space E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{0\le t\le 1} \|AS(t, 0)x\|, \quad x \in E,$$

is a Banach space. The operator valued function

$$G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. It follows from this that $AS(t) : E \to X$ is a bounded linear operator such that $AS(t)x \to 0$ as $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function, then $z(t) = \int_0^t S(t, s)x(s)ds$ defines an E-valued continuous function.

We define the operator $S(t, s)y_0 = x(t, s)$. It follows from the previous estimate that $S(t, s)$ is a bounded linear map on E. Since E is dense in X, we can extend $S(t, s)$ to X. We keep the notation $S(t, s)$ for this extension. We know that other than $\dim(X) < \infty$, the cosine function $C(t)$ cannot be compact for all $t \in \mathbb{R}$. By contrast, for the cosine functions that arise in specific applications, the sine function $S(t)$ is very often a compact operator for all $t \in \mathbb{R}$.

Theorem 2.2. *[17, Theorem 1.2]. If $S(t)$ is compact for all $t \in \mathbb{R}$, then $S(t, s)$ is also compact for all $s \le t$.*

Definition 2.3. *A function $x \in [0, b] \to X$ is said to be a mild solution of the system* (1.1)-(1.2) *if $x(t) \in D(A(t))$, for each $t \in I$ and satisfies the following integral equation*

$$\begin{aligned} x(t) =& C(t, 0)[x_0 + p(x, x')] + S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] \\ &+ \int_0^t C(t, s)g(s, x(s), x'(s))ds + \int_0^t S(t, s)f(s, x(s), x'(s))ds + \int_0^t S(t, s)Bu(s)ds, \ t \in I. \end{aligned}$$

In order to address the problem, it is convenient at this point to introduce two relevant operators and basic assumptions on these operators:

$$\Gamma_0^b = \int_0^b S(b,s)BB^*S^*(b,s)ds : X \to X,$$
$$R(\alpha, \Gamma_0^b) = (\alpha I + \Gamma_0^b)^{-1} : X \to X,$$

where B^* denotes the adjoint of B and $S^*(t)$ is the adjoint of $S(t)$. It is straightforward that the operator Γ_0^b is a linear bounded operator.

To investigate the approximate controllability of the system (1.1)-(1.2), we impose the following condition:

($\mathbf{H_0}$) $\alpha R(\alpha, \Gamma_0^b) \to 0$ as $\alpha \to 0^+$ in the strong operator topology.

In view of [23], Hypothesis ($\mathbf{H_0}$) holds if and only if the linear system

$$x''(t) = A(t)x(t) + (Bu)(t), \quad t \in [0,b], \tag{2.4}$$
$$x(0) = x_0, \quad x'(0) = y_0, \tag{2.5}$$

is approximately controllable on $[0,b]$.

Lemma 2.4 (Schauder's Fixed point theorem). *If K is a closed, bounded and convex subset of a Banach space X and $F : K \to K$ is completely continuous, then F has a fixed point in K.*

3 Approximate controllability results

In this section, first we establish a set of sufficient conditions for the approximate controllability for a class of second order neutral evolution differential equations with nonlocal conditions of the form (1.1)-(1.2) in Banach spaces by using Schauder's fixed point theorem. In order to establish the result, we need the following hypotheses:

($\mathbf{H_1}$) $S(t)$, $t > 0$ is compact.

($\mathbf{H_2}$) The function $f : I \times X \times X \to X$ satisfies the following conditions:

- (i) The function $f(t,\cdot,\cdot) : X \times X \to X$ is continuous a.e. $t \in I$.
- (ii) The function $f(t,\cdot,\cdot) : I \to X$ is strongly measurable for each $(x,y) \in X \times X$.
- (ii) For every $r > 0$, there exists a function $\lambda_r \in L'(I, R^+)$ such that

$$\sup_{\|x\|,\|y\| \le r} \|f(t,x,y)\| \le \lambda_r(t), \quad for\ a.e.\ t \in I,$$
$$\text{and} \quad \liminf_{r\to\infty} \int_0^b \frac{\lambda_r(t)}{r} dt = \delta < \infty.$$

  where $\delta > 0$ is a constant.

($\mathbf{H_3}$) The function $f : I \times X \times X \to X$ is continuous and uniformly bounded and there exists $L_f > 0$ such that $\|f(t,x,y)\| \le L_f$ for all $(t,x,y) \in I \times X \times X$.

($\mathbf{H_4}$) The function $g : I \times X \times X \to X$ satisfies the following conditions:

(i) For each $t \in I$, the function $g(t, \cdot, \cdot) : I \times X \times X \to X$ is continuous and for each $x \in X$, the function $g(\cdot, x, y) : I \times X \times X \to X$ is strongly measurable.

(ii) There exists a constants L_g, L_g^* such that

$$\begin{aligned} \|g(t, x_1, y_1) - g(t, x_2, y_2)\| \leq & L_g[\|x_1 - x_2\| + \|y_1 - y_2\|] \quad x_i, y_i \in X, i = 1, 2, \text{and} \\ \|g(t, x_1, y_1)\| \leq & L_g[\|x_1 + y_1\|] + L_g^* \end{aligned}$$

where $L_g^* = \max_{t \in I} \|g(t, 0, 0)\|$

($\mathbf{H_5}$) The functions $p, q : \mathcal{C}(I; X) \times \mathcal{C}(I; X) \to X$ are continuous and there exist positive constants L_p, L_q such that

$$\begin{aligned} \|p(x_1, y_1) - p(x_2, y_2)\| &\leq L_p(\|x_1 - x_2\| + \|y_1 - y_2\|), \\ \|q(x_1, y_1) - q(x_2, y_2)\| &\leq L_q(\|x_1 - x_2\| + \|y_1 - y_2\|), \end{aligned}$$

for every $x_1, x_2, y_1, y_2 \in \mathcal{C}(I; X)$.

It will be shown that the system (1.1)-(1.2) is approximately controllable, if for all $\alpha > 0$, there exists a continuous function $x(\cdot)$ such that

$$\begin{aligned} x(t) =& C(t, 0)[x_0 + p(x, x')] + S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] \\ &+ \int_0^t C(t, s)g(s, x(s), x'(s))ds + \int_0^t S(t, s)f(s, x(s), x'(s))ds + \int_0^t S(t, s)Bu(s, x)ds \\ x'(t) =& \frac{\partial}{\partial t}C(t, 0)[x_0 + p(x, x')] + \frac{\partial}{\partial t}S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] + g(t, x(t), x'(t)) \\ &+ \int_0^t \frac{\partial}{\partial t}C(t, s)g(s, x(s), x'(s))ds + \int_0^t \frac{\partial}{\partial t}S(t, s)f(s, x(s), x'(s))ds \\ &+ \int_0^t \frac{\partial}{\partial t}S(t, s)Bu(s, x)ds \\ u(t, x) =& B^*S^*(b, t)R(\alpha, \Gamma_0^b)p(x(\cdot)) \end{aligned}$$

where

$$\begin{aligned} p(x(\cdot)) =& x_b - C(b, 0)[x_0 + p(x, x')] - S(b, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] \\ &- \int_0^b C(b, s)g(s, x(s), x'(s))ds - \int_0^b S(b, s)f(s, x(s), x'(s))ds. \end{aligned}$$

Theorem 3.1. *Suppose that the hypotheses* ($\mathbf{H_0}$)-($\mathbf{H_6}$) *are satisfied. Assume also*

$$\begin{aligned} (M + M^*)[L_p + bL_g] + (N + N^*)[L_q + L_g + \delta] + & \frac{1}{\alpha}NM_B^2 b(N + N^*)[M(L_p + bL_g) \\ & + N(L_q + L_g + \delta)] < 1. \end{aligned} \quad (3.1)$$

where $M_B = \|B\|$. *Then system* (1.1)-(1.2) *has a solution on* I.

Proof. We consider the space $Z = \mathcal{C}(I, X) \times \mathcal{C}(I, X)$ be the space endowed with the norm of uniform convergence $\|(u, v)\|_b = \|u\|_b + \|v\|_b$. On the space Z, we consider a set Q as

$Q = \{x \in Z; x(0) = x_0 + p(x, x'), \|x\| \leq r\}$ where r is a positive constant. We define the operator $\Upsilon : Z \to Z$ by

$$\Upsilon(x, x') = (\Upsilon_1(x, x'), \Upsilon_2(x, x'))$$

where

$$\begin{aligned}\Upsilon_1(x, x') =& C(t, 0)[x_0 + p(x, x')] + S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] \\ &+ \int_0^t C(t, s)g(s, x(s), x'(s))ds + \int_0^t S(t, s)f(s, x(s), x'(s))ds + \int_0^t S(t, s)Bu(s, x)ds, \\ \Upsilon_2(x, x') =& \frac{\partial}{\partial t}C(t, 0)[x_0 + p(x, x')] + \frac{\partial}{\partial t}S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] + g(t, x(t), x'(t)) \\ &+ \int_0^t \frac{\partial}{\partial t}C(t, s)g(s, x(s), x'(s))ds + \int_0^t \frac{\partial}{\partial t}S(t, s)f(s, x(s), x'(s))ds \\ &+ \int_0^t \frac{\partial}{\partial t}S(t, s)Bu(s, x)ds.\end{aligned}$$

It will be shown that the operator Υ has a fixed point by using the following steps.

Step 1: For each positive number $r > 0$, such that $\Upsilon(Q) \subseteq Q$. If this is not true, then for each positive number r, there exists a function $(x_r(\cdot), x_r'(\cdot)) \in Q$ and $t_r \in I$, but $\Upsilon(x_r, x_r')$ does not belong to Q, i.e.,

$$\begin{aligned} r <& \|\Upsilon(x_r, x_r')(t_r)\| \\ \leq& \|\Upsilon_1(x_r, x_r')(t_r)\| + \|\Upsilon_2(x_r, x_r')(t_r)\| \\ \leq& \Big\|C(t, 0)[x_0 + p(x, x')]\Big\| + \Big\|S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))]\Big\| \\ &+ \Big\|\int_0^t C(t, s)g(s, x(s), x'(s))ds\Big\| + \Big\|\int_0^t S(t, s)f(s, x(s), x'(s))ds\Big\| \\ &+ \Big\|\int_0^t S(t, s)BB^*S^*(b, t)R(\alpha, \Gamma_0^b)\Big[x_b - C(b, 0)[x_0 + p(x, x')] - S(b, 0)[y_0 + q(x, x') \\ &- g(0, x(0), x'(0))] - \int_0^b C(b, s)g(s, x(s), x'(s))ds - \int_0^b S(b, s)f(s, x(s), x'(s))ds\Big]\Big\| \\ &+ \Big\|\frac{\partial}{\partial t}C(t, 0)[x_0 + p(x, x')]\Big\| + \Big\|\frac{\partial}{\partial t}S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))]\Big\| \\ &+ \Big\|g(t, x(t), x'(t))\Big\| + \Big\|\int_0^t \frac{\partial}{\partial t}C(t, s)g(s, x(s), x'(s))ds\Big\| + \Big\|\int_0^t \frac{\partial}{\partial t}S(t, s)f(s, x(s), x'(s))ds\Big\| \\ &+ \Big\|\int_0^t \frac{\partial}{\partial t}S(t, s)BB^*S^*(b, t)R(\alpha, \Gamma_0^b)\Big[x_b - C(b, 0)[x_0 + p(x, x')] - S(b, 0)[y_0 + q(x, x') \\ &- g(0, x(0), x'(0))] - \int_0^b C(b, s)g(s, x(s), x'(s))ds - \int_0^b S(b, s)f(s, x(s), x'(s))ds\Big]\Big\| \\ \leq& M(\|x_0\| + L_p r + \|p(0, 0)\|) + N(\|y_0\| + L_q r + \|q(0, 0)\| + L_g r + L_g^*) + bM(L_g r + L_g^*) \\ &+ N\int_0^b \lambda_r(s)ds + \frac{1}{\alpha}N^2M_B^2 b\Big[\|x_b\| + M(\|x_0\| + L_p r + \|p(0, 0)\|) + N(\|y_0\| + L_q r + \|q(0, 0)\| \\ &+ L_g r + L_g^*) + bM(L_g r + L_g^*) + N\int_0^b \lambda_r(s)ds\Big] + M^*(\|x_0\| + L_p r + \|p(0, 0)\|) \\ &+ N^*(\|y_0\| + L_q r + \|q(0, 0)\| + L_g r + L_g^*) + (L_g r + L_g^*) + bM^*(L_g r + L_g^*) + N^*\int_0^b \lambda_r(s)ds \end{aligned}$$

$$+\frac{1}{\alpha}NN^*M_B^2b\Big[\|x_b\|+M(\|x_0\|+L_pr+\|p(0,0)\|)+N(\|y_0\|+L_qr+\|q(0,0)\|+L_gr+L_g^*)$$
$$+bM(L_gr+L_g^*)+N\int_0^b\lambda_r(s)ds\Big]$$

Dividing both sides of the equation by r and taking the limit as $n\to\infty$, we get

$$(M+M^*)[L_p+bL_g]+(N+N^*)[L_q+L_g+\delta]+L_g+\frac{1}{\alpha}NM_B^2b(N+N^*)[M(L_p+bL_g)$$
$$+N(L_q+L_g+\delta)]\geq 1.$$

This contradicts with the condition (3.1). Hence, for some $r>0$, $\overline{\Upsilon}(\mathcal{B}_r)\subseteq\mathcal{B}_r$.

Step 2: The set $\Pi(t)=\{\Upsilon(x,x')(t)\in(\mathcal{C}[0,b]\times\mathcal{C}[0,b],X)|x\in Q\}$ is an equicontinuous family of function on $[0,b]$. For $0<t_1<t_2\leq b$ and $\varepsilon<0$, then

$$\|\Upsilon_1(x,x')(t_1)-\Upsilon_1(x,x')(t_2)\|$$
$$\leq\|C(t_1,0)-C(t_2,0)\|\|[x_0+p(x,x')]\|+\|S(t_1,0)-S(t_2,0)\|\|[y_0+q(x,x')$$
$$-g(0,x(0),x'(0))]\|+\left\|\int_{t_1}^{t_2}C(t_2,s)g(s,x(s),x'(s))ds\right\|$$
$$+\left\|\int_0^{t_1-\epsilon}[C(t_1,s)-C(t_2,s)]g(s,x(s),x'(s))ds\right\|$$
$$+\left\|\int_{t_1-\epsilon}^{t_1}[C(t_1,s)-C(t_2,s)]g(s,x(s),x'(s))ds\right\|+\left\|\int_{t_1}^{t_2}S(t_2,s)f(s,x(s),x'(s))ds\right\|$$
$$+\left\|\int_0^{t_1-\epsilon}[S(t_1,s)-S(t_2,s)]f(s,x(s),x'(s))ds\right\|$$
$$+\left\|\int_{t_1-\epsilon}^{t_1}[S(t_1,s)-S(t_2,s)]f(s,x(s),x'(s))ds\right\|+\left\|\int_{t_1}^{t_2}S(t_2,\eta)Bu(\eta,x)d\eta\right\|$$
$$+\left\|\int_0^{t_1-\epsilon}[S(t_1,\eta)-S(t_2,\eta)]Bu(\eta,x)d\eta\right\|$$
$$+\left\|\int_{t_1-\epsilon}^{t_1}[S(t_1,\eta)-S(t_2,\eta)]Bu(\eta,x)d\eta\right\|$$
$$\leq\|C(t_1,0)-C(t_2,0)\|\|[x_0+p(x,x')]\|+\|S(t_1,0)-S(t_2,0)\|\|[y_0+q(x,x')$$
$$-g(0,x(0),x'(0))]\|+M\int_{t_1}^{t_2}(L_g\|x(s)+x'(s)\|+L_g^*)ds$$
$$+\int_0^{t_1-\epsilon}[C(t_1,s)-C(t_2,s)](L_g\|x(s)+x'(s)\|+L_g^*)ds$$
$$+\int_{t_1-\epsilon}^{t_1}[C(t_1,s)-C(t_2,s)](L_g\|x(s)+x'(s)\|+L_g^*)ds+N\int_{t_1}^{t_2}\lambda_r(s)ds$$
$$+\int_0^{t_1-\epsilon}[S(t_1,s)-S(t_2,s)]\lambda_r(s)ds+\int_{t_1-\epsilon}^{t_1}[S(t_1,s)-S(t_2,s)]\lambda_r(s)ds$$
$$+NM_B\int_{t_1}^{t_2}\|u(\eta,x)\|d\eta+M_B\int_0^{t_1-\epsilon}[S(t_1,\eta)-S(t_2,\eta)]\|u(\eta,x)\|d\eta$$
$$+M_B\int_{t_1-\epsilon}^{t_1}[S(t_1,\eta)-S(t_2,\eta)]\|u(\eta,x)\|d\eta.$$

Similarly, we find that

$$
\begin{aligned}
&\|\Upsilon_2(x,x')(t_1)-\Upsilon_2(x,x')(t_2)\| \\
&\quad\leq \left\|\frac{\partial}{\partial t_1}C(t_1,0)-\frac{\partial}{\partial t_2}C(t_2,0)\right\|\left\|x_0+p(x,x')\right\|+\left\|\frac{\partial}{\partial t_1}S(t_1,0)-\frac{\partial}{\partial t_2}S(t_2,0)\right\| \\
&\qquad\times\left\|y_0+q(x,x')-g(0,x(0),x'(0))\right\|+\left\|g\left((t,x(t_1),x'(t_1))-g(t,x(t_2),x'(t_2))\right)\right\| \\
&\qquad+\left\|\int_{t_1}^{t_2}\frac{\partial}{\partial t_2}C(t_2,s)g(s,x(s),x'(s))ds\right\|+\left\|\int_0^{t_1-\epsilon}\left[\frac{\partial}{\partial t_1}C(t_1,s)-\frac{\partial}{\partial t_2}C(t_2,s)\right]\right. \\
&\qquad\times g(s,x(s),x'(s))ds\Big\|+\left\|\int_{t_1-\epsilon}^{t_1}\left[\frac{\partial}{\partial t_1}C(t_1,s)-\frac{\partial}{\partial t_2}C(t_2,s)\right]g(s,x(s),x'(s))ds\right\| \\
&\qquad+\left\|\int_{t_1}^{t_2}\frac{\partial}{\partial t_2}S(t_2,s)f(s,x(s),x'(s))ds\right\|+\left\|\int_0^{t_1-\epsilon}\left[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\right]\right. \\
&\qquad\times f(s,x(s),x'(s))ds\Big\|+\left\|\int_{t_1-\epsilon}^{t_1}\left[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\right]f(s,x(s),x'(s))ds\right\| \\
&\qquad+\left\|\int_{t_1}^{t_2}\frac{\partial}{\partial t_2}S(t_2,\eta)Bu(\eta,x)d\eta\right\|+\left\|\int_0^{t_1-\epsilon}\left[\frac{\partial}{\partial t_1}S(t_1,\eta)-\frac{\partial}{\partial t_2}S(t_2,\eta)\right]\right. \\
&\qquad\times Bu(\eta,x)d\eta\Big\|+\left\|\int_{t_1-\epsilon}^{t_1}\left[\frac{\partial}{\partial t_1}S(t_1,\eta)-\frac{\partial}{\partial t_2}S(t_2,\eta)\right]Bu(\eta,x)d\eta\right\| \\
&\quad\leq \left\|\frac{\partial}{\partial t_1}C(t_1,0)-\frac{\partial}{\partial t_2}C(t_2,0)\right\|\left\|[x_0+p(x,x')]\right\|+\left\|\frac{\partial}{\partial t_1}S(t_1,0)-\frac{\partial}{\partial t_2}S(t_2,0)\right\| \\
&\qquad\times\left\|y_0+q(x,x')-g(0,x(0),x'(0))\right\|+\left\|g\left((t,x(t_1),x'(t_1))-g(t,x(t_2),x'(t_2))\right)\right\| \\
&\qquad+M^*\int_{t_1}^{t_2}(L_g\|x(s)+x'(s)\|+L_g^*)ds+\int_0^{t_1-\epsilon}\left[\frac{\partial}{\partial t_1}C(t_1,s)-\frac{\partial}{\partial t_2}C(t_2,s)\right] \\
&\qquad\times(L_g\|x(s)+x'(s)\|+L_g^*)ds+\int_{t_1-\epsilon}^{t_1}\left[\frac{\partial}{\partial t_1}C(t_1,s)-\frac{\partial}{\partial t_2}C(t_2,s)\right](L_g\|x(s)+x'(s)\| \\
&\qquad+L_g^*)ds+N^*\int_{t_1}^{t_2}\lambda_r(s)ds+\int_0^{t_1-\epsilon}\left[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\right]\lambda_r(s)ds \\
&\qquad+\int_{t_1-\epsilon}^{t_1}\left[\frac{\partial}{\partial t_1}S(t_1,s)-\frac{\partial}{\partial t_2}S(t_2,s)\right]\lambda_r(s)ds+N^*M_B\int_{t_1}^{t_2}\|u(\eta,x)\|d\eta \\
&\qquad+M_B\int_0^{t_1-\epsilon}\left[\frac{\partial}{\partial t_1}S(t_1,\eta)-\frac{\partial}{\partial t_2}S(t_2,\eta)\right]\|u(\eta,x)\|d\eta \\
&\qquad+M_B\int_{t_1-\epsilon}^{t_1}\left[\frac{\partial}{\partial t_1}S(t_1,\eta)-\frac{\partial}{\partial t_2}S(t_2,\eta)\right]\|u(\eta,x)\|d\eta
\end{aligned}
$$

The right-hand side of the above inequality tends to zero independently of $x\in Q$ as $(t_1-t_2)\to 0$ and ε sufficiently small, since the compactness of the evolution operator $C(t,s), S(t,s)$ implies the continuity in the uniform operator topology. Thus $\Upsilon(x,x')(t)$ sends Q into equicontinuous family of functions.

Step 3. The set $\Pi(t)=\left\{\Upsilon(x,x')(t): x\in Q\right\}$ is relatively compact in X for every $t\in I$. The case $t=0$ is trivial. Clearly, $\Pi(0)=\left\{(\Upsilon x)(0):(x,x')(\cdot)\in Q\right\}=x_0+p(x,x')$ is compact in X.

So, let $t \in (0, b]$ be fixed and ε a real number satisfying $0 < \varepsilon < t \leq b$. We define $x \in Q$,

$$\begin{aligned}\Upsilon_1^{\varepsilon}(x, x')(t) =& C(t, 0)[x_0 + p(x, x')] + S(t, 0)[y_0 + q(x, x') - g(0, x(0), x'(0))] \\ &+ \int_0^{t-\varepsilon} C(t, s)g(s, x(s), x'(s))ds + \int_0^{t-\varepsilon} S(t, s)f(s, x(s), x'(s))ds \\ &+ \int_0^{t-\varepsilon} S(t, \eta)Bu(\eta, x)d\eta\end{aligned}$$

Since $S(t, s)$ is a compact operator, the set $\Pi_{\varepsilon}(t) = \left\{\Upsilon_1^{\varepsilon}(x, x')(t) : (x, x')(\cdot) \in Q\right\}$ is relatively compact in X for each ε, $0 < \varepsilon < t$. Moreover, for each $0 < \varepsilon < t$, we have

$$\begin{aligned}\|\Upsilon_1(x, x')(t) - \Upsilon_1^{\varepsilon}(x, x')(t)\| \leq& \int_{t-\varepsilon}^{t} \|C(t, s)g(s, x(s), x'(s))\|ds + \int_{t-\varepsilon}^{t} \|S(t, s)f(s, x(s), x'(s))\|ds \\ &+ \int_{t-\varepsilon}^{t} \|S(t, \eta)Bu(\eta, x)\|d\eta \\ \leq& M\int_{t-\varepsilon}^{t} (L_g\|x(s) + x'(s)\| + L_g^*)ds + N\int_{t-\varepsilon}^{t} \lambda_r(s)ds \\ &+ NM_B\int_{t-\varepsilon}^{t} \|u(\eta, x)\|d\eta, \qquad t \in I.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\Upsilon_2(x, x')(t) - \Upsilon_2^{\varepsilon}(x, x')(t)\| \leq& \|g(t, x(t), x'(t)) - C(\varepsilon)g(t-\varepsilon, x(t-\varepsilon), x'(t-\varepsilon))\| \\ &+ \int_{t-\varepsilon}^{t} \|\frac{\partial}{\partial t}C(t, s)g(s, x(s), x'(s))\|ds \\ &+ \int_{t-\varepsilon}^{t} \|\frac{\partial}{\partial t}S(t, s)f(s, x(s), x'(s))\|ds \\ &+ \int_{t-\varepsilon}^{t} \|\frac{\partial}{\partial t}S(t, \eta)Bu(\eta, x)\|d\eta \\ \leq& \|g(t, x(t), x'(t)) - C(\varepsilon)g(t-\varepsilon, x(t-\varepsilon), x'(t-\varepsilon))\| \\ &+ M^*\int_{t-\varepsilon}^{t} (L_g\|x(s) + x'(s)\| + L_g^*)ds + N^*\int_{t-\varepsilon}^{t} \lambda_r(s)ds \\ &+ N^*M_B\int_{t-\varepsilon}^{t} \|u(\eta, x)\|d\eta, \qquad t \in I.\end{aligned}$$

There exists relatively compact sets arbitrarily close to the set $\Pi(t)$, for each $t \in (0, b]$. Hence $\Pi(t), t \in (0, b]$ is relatively compact in X. Since it is compact at $t = 0$, we have $\Pi(t)$ is relatively compact in X for all $t \in [0, b]$.

Step 4. The map $\Upsilon(\cdot)$ is continuous on Q.

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence in Q and we can find a number $q > 0$ in such a way that $\|x_n(t)\| \leq q$, for all n and a.e. $t \in I$. So, $x_n = B_q = \{x \in Q : \|x\| \leq q\} \subseteq Q$ and $x \in B_q$.

$$\begin{aligned}g(s, x_n(s), x_n'(s)) \to& g(s, x(s), x'(s)), \\ f(s, x_n(s), x_n'(s)) \to& f(s, x(s), x'(s)), \\ u(s, z_n + y) \to& u(s, z + y),\end{aligned}$$

for every $t \in I$ and by (H_2),

$$\|g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))\| \leq 2(L_g q' + L_g^*)$$
$$\|f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\| \leq 2\lambda_q(s),$$
$$\|u(s,z^n+y) - u(s,z+y)\| \leq \frac{2}{\alpha} L_u,$$

where

$$L_u = NM_B\Big[\|x_b\| + M\|x_0 + p(x,x')\| + N\|y_0 + q(x,x') - g(0,x(0),x'(0))\| + M\int_0^b (L_g q + L_g^*)ds + N\int_0^b \lambda_q(s)ds\Big].$$

By the above steps and Lebesgue Dominated Convergence theorem, we get

$$\begin{aligned}\|\Upsilon_1(x_n,x_n')(t) - \Upsilon_1(x,x')(t)\| \leq & \sup_{t\in I}\Big[\Big\|\int_0^t C(t,s)[g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))]ds \\ & + \int_0^t S(t,s)[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))]ds \\ & + \int_0^t S(t,\eta)[Bu(\eta,x_n) - Bu(\eta,x)]d\eta\Big\|\Big] \\ \leq & M\int_0^b \|g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))\|ds \\ & + N\int_0^b \|f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\|ds \\ & + NM_B\int_0^b \|u(\eta,x_n) - u(\eta,x)\|d\eta \\ \to & 0 \text{ as } n\to\infty,\end{aligned}$$

and

$$\begin{aligned}\|\Upsilon_2(x_n,x_n')(t) - \Upsilon_2(x,x')(t)\| \leq & \sup_{t\in I}\Big[\Big\|[g(t,x_n(t),x_n'(t)) - g(t,x(t),x(t))] \\ & + \int_0^t \frac{\partial}{\partial t}C(t,s)[g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))]ds \\ & + \int_0^t \frac{\partial}{\partial t}S(t,s)[f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))]ds \\ & + \int_0^t \frac{\partial}{\partial t}S(t,\eta)[Bu(\eta,x_n) - Bu(\eta,x)]d\eta\Big\|\Big] \\ \leq & \|[g(t,x_n(t),x_n'(t)) - g(t,x(t),x(t))]\| \\ & + M^*\int_0^b \|g(s,x_n(s),x_n'(s)) - g(s,x(s),x'(s))\|ds \\ & + N^*\int_0^b \|f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\|ds \\ & + N^*M_B\int_0^b \|u(\eta,x_n) - u(\eta,x)\|d\eta \\ \to & 0 \text{ as } n\to\infty,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|\Upsilon(x_n, x_n')(t) - \Upsilon(x, x')(t)\| = 0.$$

Thus, Υ is continuous. Hence $\Upsilon(\cdot)$ is equicontinuous and also bounded. By the Ascoli-Arzela theorem, Υ is relatively compact in $\mathcal{C}([0,b]\times[0,b],X)$. Also, Υ is continuous on $\mathcal{C}([0,b]\times[0,b],X)$. Hence, Υ is completely continuous operator on $\mathcal{C}([0,b]\times[0,b],X)$. Thus from the Schauder's fixed point theorem, Υ has a fixed point, which is a mild solution of system (1.1)-(1.2). □

Definition 3.2. *The control system* (1.1) *is said to be approximately controllable on* I *if* $\overline{R(b,x_0)} = X$, *where* $R(b,x_0) = \{x_b(x_0;u) : u(\cdot) \in L^1(I,U)\}$ *is called the reachable set of system* (1.1) *at terminal time* b *and its closure in* X *is denoted by* $\overline{R(b,x_0)}$; *Let* $x_b(x_0,u)$ *be the state value of* (1.1) *at terminal time* b *corresponding to the control* u *and the initial value* $x_0 \in X$.

Frankly speaking, by using the control function u, from any given initial point $x_0 \in X$ we can move the system with the trajectory as close as possible to any other final point $x_b \in X$.

Theorem 3.3. *Suppose that the assumptions* ($\mathbf{H_0}$)-($\mathbf{H_5}$) *hold. Then the nonlinear second order differential equation* (1.1)-(1.2) *is approximately controllable on* I.

Proof. Let $\widehat{x}_\alpha(\cdot)$ be a fixed point of Υ in Q. By Theorem 3.1, any fixed point of Υ is a mild solution of (1.1)-(1.2) under the control

$$\widehat{u}_\alpha(t) = B^*S^*(b,t)R(\alpha,\Gamma_0^b)p(\widehat{x}_\alpha)$$

and satisfies the following inequality

$$\widehat{x}_\alpha(b) = x_b + \alpha R(\alpha,\Gamma_0^b)p(\widehat{x}_\alpha). \tag{3.2}$$

By the condition ($\mathbf{H_3}$) and ($\mathbf{H_4}$),

$$\int_0^b \|g(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s))\|^2 ds \le bL_g^2,$$
$$\int_0^b \|f(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s))\|^2 ds \le bL_f^2.$$

Moreover by assumption on f and Dunford-Pettis theorem, we have that $g(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s))$ and $f(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s))$ are weakly compact in $L^1(I,X)$, so there is a subsequence, still denoted by $g(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s))$ and $f(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s))$, that converges weakly to, say $g(s)$ and $f(s)$, in $L^1(I,X)$. Define

$$w = x_b - C(b,0)[x_0 + p(x,x')] - S(b,0)[y_0 + q(x,x')] - \int_0^b C(b,s)g(s)ds - \int_0^b S(b,s)f(s)ds$$

Now we have,

$$\begin{aligned}\|p(\widehat{x}_\alpha) - w\| =& \left\| \int_0^b C(b,s)[g(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s)) - g(s)]ds \right\| \\ &+ \left\| \int_0^b S(b,s)[f(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s)) - f(s)]ds \right\| \\ \le& \sup_{t\in I} \left[\left\| \int_0^t C(b,s)[g(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s)) - g(s)]ds \right\| \right. \\ &+ \left. \left\| \int_0^t S(b,s)[f(s,\widehat{x}_\alpha(s),\widehat{x}'_\alpha(s)) - f(s)]ds \right\| \right]\end{aligned} \tag{3.3}$$

By using infinite-dimensional version of the Ascoli-Arzela theorem, we can show that an operator $l(\cdot) \to \int_0^{\cdot} S(\cdot, s)l(s)ds : L^1(I, X) \to C(I, X)$ is compact. Therefore, we obtain that $\|p(\widehat{x}_\alpha) - w\| \to 0$ as $\alpha \to 0^+$. Moreover, from (3.2) we get

$$\begin{aligned} \|\widehat{x}_\alpha(b) - x_b\| \leq & \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|\alpha R(\alpha, \Gamma_0^b)\| \|p(\widehat{x}_\alpha) - w\| \\ \leq & \|\alpha R(\alpha, \Gamma_0^b)(w)\| + \|p(\widehat{x}_\alpha) - w\|. \end{aligned}$$

It follows from assumption $(\mathbf{H_0})$ and the estimation (3.3) that $\|\widehat{x}_\alpha(b) - x_b\| \to 0$ as $\alpha \to 0^+$. This proves the approximate controllability of differential equation (1.1)-(1.2). □

4 Second order impulsive evolution differential equation

Various evolutionary processes from fields such as physics, population dynamics, aeronautics, economics and engineering are characterized by the fact that they undergo abrupt changes of state at certain moments of time between intervals of continuous evolution. Since the duration of these changes are often negligible compared to the total period of time, such changes can be reasonably well approximated in the form of impulses. These process tend to more suitably modeled by impulsive differential equations. For more details on this theory, we refer the monographs of [3, 6, 9, 20, 22, 29–31, 37, 38].

Inspired by the above works, in this paper, we establish sufficient conditions for the approximate controllability for a class of second-order neutral impulsive evolution differential equations with nonlocal conditions in Banach spaces of the form

$$\frac{d}{dt}[x'(t) - g\left(t, x(t), x'(t)\right)] = A(t)x(t) + f\left(t, x(t), x'(t)\right) + Bu(t), \quad t \in I = [0, b], \tag{4.1}$$

$$x(0) = x_0 + p(x, x'), \quad x'(0) = y_0 + q(x, x'), \tag{4.2}$$

$$\Delta x(t_i) = I_i(x(t_i), x'(t_i)), \quad i = 1, 2, ...n, \tag{4.3}$$

$$\Delta x'(t_i) = J_i(x(t_i), x'(t_i)), \quad i = 1, 2, ...n. \tag{4.4}$$

In this equation, $A(t) : D(A(t)) \subseteq X \to X$ is a closed linear operator on a Banach space X with norm $\|\cdot\|$. Here U is a Banach space and B is a bounded linear operator from U to X and the functions. Also, $f : I \times X \times X \to X$, $p, q : \mathcal{PC} \times \mathcal{PC} \to X$ are the appropriate functions defined later. Similarly, the functions $I_i(\cdot) : X \times X \to X$, $J_i(\cdot) : X \times X \to X$ defined later in the preliminaries section. The symbol $\Delta\xi(t)$ represents the jump of the function $\xi(\cdot)$ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$.

To consider the impulsive conditions $(4.3) - (4.4)$, it is necessary to introduce some additional concepts and notations.

To simplify the notations, we put $t_0 = 0, t_{n+1} = b$. Now we define the space $\mathcal{PC}([0, b], X)$ formed by the functions $x : [0, b] \to X$ such that $x(\cdot)$ is continuous at $t \neq t_i$, $x(t_i^-) = x(t_i)$ and $x(t_i^+)$ exists for all $i = 1, 2, ..., n$, endowed with the uniform norm $\|\cdot\|_{\mathcal{PC}}$ which is defined by $\|x\|_{\mathcal{PC}} = \sup_{s \in I} \|x(s)\|$. It is clear that $(\mathcal{PC}, \|\cdot\|_{\mathcal{PC}})$ is a Banach space. Similarly, $\mathcal{PC}'$ will be the space of the functions $x \in \mathcal{PC}$ such that x is continuously differentiable on $I \backslash \{t_i : i = 1, 2, ..., n\}$ and the lateral derivatives $x'_R(t) = \lim_{s \to 0^+} \frac{x(t+s) - x(t^+)}{s}$, $x'_L(t) = \lim_{s \to 0^-} \frac{x(t+s) - x(t^-)}{s}$ are the continuous functions on $[t_i, t_{i+1})$and $(t_i, t_{i+1}]$ respectively. Next, for $x \in \mathcal{PC}'$, we represent by $x'(t)$ the left derivative at $t \in (0, b]$ and by $x'(0)$ the right derivative at zero. It is easy to see that the space $\mathcal{PC}'$ endowed with the norm $\|x\|_{\mathcal{PC}'} = \|x\|_{\mathcal{PC}} + \|x'\|_{\mathcal{PC}}$ is a Banach space.

$(\mathbf{H_6})$ There are positive constants L_{I_i}, L_{J_i} such that

$$\|I_i(x_1, y_1) - I_i(x_2, y_2)\| \leq L_{I_i}[\|x_1 - x_2\| + \|y_1 - y_2\|],$$
$$\|J_i(x_1, y_1) - J_i(x_2, y_2)\| \leq L_{J_i}[\|x_1 - x_2\| + \|y_1 - y_2\|], \forall x_i, y_i \in X, i = 1, 2.$$

Definition 4.1. *A function $x \in [0, b] \to X$ is said to be a mild solution of the system (4.1)-(4.4) if $x(t) \in D(A(t))$, for each $t \in I$ and satisfies the following integral equation*

$$\begin{aligned} x(t) =& C(t,0)[x_0 + p(x,x')] + S(t,0)[y_0 + q(x,x') - g(0,x(0),x'(0))] \\ &+ \int_0^t C(t,s)g(s,x(s),x'(s))ds + \int_0^t S(t,s)f(s,x(s),x'(s))ds + \int_0^t S(t,s)Bu(s)ds \\ &+ \sum_{0<t_i<t} C(t,t_i)I_i(x(t_i),x'(t_i)) + \sum_{0<t_i<t} S(t,t_i)J_i(x(t_i),x'(t_i)), \ t \in I. \end{aligned}$$

Theorem 4.2. *Assume that the assumptions of Theorem 3.1 are satisfied. Further, if the hypothesis $(\mathbf{H_6})$ is satisfied, then the system (4.1)-(4.4) is approximately controllable on I provided that*

$$\begin{aligned} &(M + M^*)\Big[L_p + bL_g + \sum_{i=1}^n L_{I_i}\Big] + (N + N^*)\Big[L_q + L_g + \delta + \sum_{i=1}^n L_{J_i}\Big] + L_g \\ &\quad + \frac{1}{\alpha} N M_B^2 b(N + N^*)\Big[M\Big(L_p + bL_g + \sum_{i=1}^n L_{I_i}\Big) + N\Big(L_q + L_g + \delta + \sum_{i=1}^n L_{J_i}\Big)\Big] < 1, \end{aligned}$$

where $M_B = \|B\|$.

Proof. We define the operator $\widehat{\Upsilon} : Z \to Z$ by

$$\widehat{\Upsilon}(x, x') = (\widehat{\Upsilon}_1(x, x'), \widehat{\Upsilon}_2(x, x'))$$

where

$$\begin{aligned} \widehat{\Upsilon}_1(x,x') =& C(t,0)[x_0 + p(x,x')] + S(t,0)[y_0 + q(x,x') - g(0,x(0),x'(0))] \\ &+ \int_0^t C(t,s)g(s,x(s),x'(s))ds + \int_0^t S(t,s)f(s,x(s),x'(s))ds + \int_0^t S(t,s)Bu(s,x)ds \\ &+ \sum_{0<t_i<t} C(t,t_i)I_i(x(t_i),x'(t_i)) + \sum_{0<t_i<t} S(t,t_i)J_i(x(t_i),x'(t_i)), \\ \widehat{\Upsilon}_2(x,x') =& \frac{\partial}{\partial t}C(t,0)[x_0 + p(x,x')] + \frac{\partial}{\partial t}S(t,0)[y_0 + q(x,x') - g(0,x(0),x'(0))] + g(t,x(t),x'(t)) \\ &+ \int_0^t \frac{\partial}{\partial t}C(t,s)g(s,x(s),x'(s)) + \int_0^t \frac{\partial}{\partial t}S(t,s)f(s,x(s),x'(s))ds \\ &+ \int_0^t \frac{\partial}{\partial t}S(t,s)Bu(s,x)ds + \sum_{0<t_i<t} \frac{\partial}{\partial t}C(t,t_i)I_i(x(t_i),x'(t_i)) \\ &+ \sum_{0<t_i<t} \frac{\partial}{\partial t}S(t,t_i)J_i(x(t_i),x'(t_i)) \\ u(t,x) =& B^*S^*(b,t)R(\alpha,\Gamma_0^b)p(x(\cdot)), \end{aligned}$$

where

$$p(x(\cdot)) = x_b - C(b,0)[x_0 + p(x,x')] - S(b,0)[y_0 + q(x,x') - g(0,x(0),x'(0))] - \int_0^b C(b,s)g(s,x(s),x'(s)) - \int_0^b S(b,s)f(s,x(s),x'(s))ds - \sum_{0<t_i<b} C(b,t_i)I_i(x(t_i),x'(t_i)) - \sum_{0<t_i<b} S(b,t_i)J_i(x(t_i),x'(t_i))$$

By following the techniques in Theorem 3.1, for all $\alpha > 0$, the operator $\widehat{\Upsilon}$ has a fixed point. Hence, we can show that the system (4.1)-(4.4) is approximately controllable with the help of Theorem 3.3. Since the proof of this theorem is similar to that of Theorem 3.1 and Theorem 3.3, we can skip this section. □

5 An application

In this section, we apply our abstract results to a concrete impulsive partial differential equation. The following technical framework is needed to prove our results.

Here we consider $A(t) = A + \widetilde{B}(t)$ where A is the infinitesimal generator of a cosine function $C_0(t)$ with associated sine function $S_0(t)$, and $\widetilde{B}(t) : D(\widetilde{B}(t)) \to X$ is a closed linear operator with $D \subseteq D(\widetilde{B}(t))$ for all $t \in I$. In this application, we use the space $X = L^2(\mathbb{T},\mathbb{C})$, where the group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi Z$. We will use the identification between functions on $\mathbb{T}$ and 2π-periodic 2-integrable functions from $\mathbb{R}$ into $\mathbb{C}$. Similarly, $H^2(\mathbb{T},\mathbb{C})$ denotes the Sobolev space of 2π-periodic functions $x : \mathbb{R} \to \mathbb{C}$ such that $x'' \in L^2(\mathbb{T},\mathbb{C})$.

We introduce the operator $Ax(\xi) = x''(\xi)$ with domain $D(A) = H^2(\mathbb{T},\mathbb{C})$, where A is the infinitesimal generator of a strongly continuous cosine family $C_0(t)$ on X. Furthermore, A has a discrete spectrum, the eigen values are $-n^2$ for $n \in \mathbb{Z}$, with associated eigenvectors

$$z_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, \quad n \in Z$$

and the following properties hold:

(a) The set $\{z_n : n \in Z\}$ is an orthonormal basis of X. In particular,

$$Ax = -\sum_{n=1}^{\infty} n^2 \langle x, w_n \rangle w_n, \quad \text{for } x \in D(A).$$

(b) For $x \in X$, the cosine function $C_0(t)$ is given by

$$C_0(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, w_n \rangle w_n, \quad t \in \mathbb{R},$$

with associated sine function

$$S_0(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, w_n \rangle w_n, \quad t \in \mathbb{R}.$$

It is clear that $\|C_0(t)\| \le 1$ for all $t \in \mathbb{R}$. Thus, $C_0(\cdot)$ is uniformly bounded on $\mathbb{R}$. Consider the second-order partial differential problem with control

$$\frac{\partial}{\partial t}\Big[\frac{\partial}{\partial t}u(t,\xi) + G(t,u(t,\xi),\frac{\partial}{\partial t}u(t,\xi))\Big] = \frac{\partial^2}{\partial \xi^2}u(t,\xi) + b(t)\frac{\partial}{\partial t}u(t,\xi) + \mu(t,\xi) + F(t,u(t,\xi),\frac{\partial}{\partial t}u(t,\xi)), \tag{5.1}$$

for $t \in I$, $0 \leq \xi \leq \pi$, subject to the initial conditions

$$u(t,0) = u(t,\pi) = 0, \quad t \in I, \tag{5.2}$$

$$u(0,\xi) = z_0(\xi) + \sum_{i=1}^{n} \alpha_i u(t_i,\xi), \tag{5.3}$$

$$\frac{\partial}{\partial t} u(0,\xi) = z_1(\xi) + \sum_{i=1}^{n} \beta_i u(s_i,\xi), \quad 0 \leq \xi \leq \pi, \tag{5.4}$$

where $z_0, z_1 \in X$, $a_i, \widetilde{a_i} \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $0 < t_i, s_i < \pi$, α_i, β_i are prefixed numbers, $b : [0,\infty) \to \mathbb{R}$, $G, F : I \times X \times X \to X$ are continuous functions. We fix $a > 0$ and set $\beta = \sup_{0 \leq t \leq a} |b(t)|$.

We take $\widetilde{B}(t)x(\xi) = b(t)x'(\xi)$ defined on $H^1(\mathbb{T},\mathbb{C})$. We have to show that the control function μ which steers (5.1) from any specified initial state to the final state in a Banach space X.

Assume that the bounded linear operator $B : U \subset I \to X$ is define by

$$B(u(t))(\xi) = \mu(t,\xi), \quad \xi \in [0,\pi],$$

where $\mu : I \times [0,\pi] \to [0,\pi]$ is continuous. Define the operators $f, g : I \times X \times X \to X$, $p, q : \mathcal{C} \times \mathcal{C} \to X$ which are continuous by

$$g(t,x,y)(\xi) = G(t,x(\xi),y(\xi)),$$
$$f(t,x,y)(\xi) = F(t,x(\xi),y(\xi)),$$
$$p(x,y)(\xi) = \sum_{i=1}^{n} \alpha_i x(t_i,\xi),$$
$$q(x,y)(\xi) = \sum_{i=1}^{n} \beta_i x(t_i,\xi),$$

Assume these functions satisfy the requirement of hypotheses. From the above choices of the functions and evolution operator $A(t)$ with $B = I$, the system (5.1)-(5.4) can be formulated as an abstract second order semilinear system (1.1)-(1.2) in X. Since all hypotheses of Theorem 3.3 are satisfied, approximate controllability of system (5.1)-(5.4) on I follows from Theorem 3.3.

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