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AN ANALOGY OF THE SAINT- VENANT'S PRINCIPLE FOR SOLUTIONS OF THE THIRD ORDER PSEUDOELLIPTICAL EQUATIONS

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Abstract: In the paper energy estimates for solutions of third order equations considering transformation's character of the body form were established. In consequence of this estimate, the uniqueness theorems were obtained for solutions of the first boundary value problem for third order equations in unlimited domains.
***Keywords*:** equations of the pseudo elliptical type, analogy of the Saint-Venant's principle, uniqueness theorem, generalized solution, unlimited domain, energy estimate, equations of third order.

INTRODUCTION

The Saint-Venant principle (see [4], [14]) is expressed in the planar theory of elasticity as a prior estimate for a solution of a biharmonic equation satisfying homogeneous boundary conditions of the first boundary value problem in the part of the domain boundary. Such energetic estimates were obtained first in [7], [5]. These estimates do not take into account character of transformation of the body form at moving off from those part of the bound where exterior forces are applied. In [10], another proof of the Saint-Venant's principle in the planar theory of elasticity was given. The energetic estimate obtained in this connection considered character of transformation of the body form. As a corollary of this estimate, the uniqueness theorem for the first boundary value problem of the planar theory of elasticity in unlimited domains and also Pharagmen-Lindelof type theorems were obtained. Some Pharagmen-Lindelof type theorems were proved for equations of the theory of elasticity in [16] and for elliptic equations of higher order in [2]-[9]. The Saint-Venant principle for a cylindrical body was proved in [15]. An analog of the Saint-Venant principle, uniqueness theorems in unlimited domains, and Pharagmen-Lindelof type theorems were obtained for the system of equations of the theory of elasticity in [12] in the case of space with boundary conditions of the first boundary value problem. For the mixed problems similar results were derived in [13].

In the present paper, the analogy of the Saint-Venant principle is established for the generalized solution of the third order pseudo elliptical type equation. Also uniqueness theorems are obtained for solutions of the first boundary value problem in classes of functions increasing in infinity depending on the geometric characteristics of the domain $Q = D \times \Omega \times (0,T)$, were $D \subset \mathbb{R}_{+}^{n} = \{x : x_1 > 0\}$, Ω is bounded domain. Boundary value problems for the third order pseudo elliptical type equations in bounded domains were considered in [8].

We shall note else work [6], [1], which by means of principle Saint-Venant's is studied asymptotic characteristic of the solutions of the third order equations of the composite type and dynamic systems.

- **1. Notations and formulation of the problem**

Consider in the unlimited domain Q the equation

$$L_0 lu + L_1 u + Mu = f(x,y,t)$$

where

$$lu = u_t + \alpha^k(x)u_{x_k} + \alpha_0(x)u, \quad L_1 u = b^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i},$$

$$L_o u = u_t - a^{ij}(x)u_{x_i x_j} + a^i(x)u_{x_i} + a_0(x)u,$$

$$Mu = c^{pq}(x)u_{y_p y_q} + c^p(x)u_{y_p} + c_0(x)u.$$

We suppose here and later on that the summation is carried out by repeating indexes, all coefficients in (1) and their derivatives are bounded and measurable in any finite sub domain of the domain Q. Also we suppose that boundary of Q is smooth or piecewise-smooth. We assume that the operators L_o, M are uniformly elliptic, i.e.

$$a^{ij} = a^{ji}, \quad \lambda_0 |\xi|^2 \le a^{ij}\xi_i\xi_j \le \lambda_1 |\xi|^2, \quad \forall (x,y,t) \in Q \cup \partial Q, \quad \forall \xi \in \mathbb{R}^{n+m+1}$$

$$c^{pq} = c^{qp}, \quad \mu_0 |\xi|^2 \le a^{ij}\xi_i\xi_j \le \mu_1 |\xi|^2, \quad \forall (x,y,t) \in Q \cup \partial Q, \quad \forall \xi \in \mathbb{R}^{n+m+1}. \qquad (2)$$

Let $G = D \times \Omega$ and $\nu(x) = (\nu_{x_1},\dots,\nu_{x_n},\nu_{y_1},\dots,\nu_{y_m},\nu_t)$ is a vector of the inner normal of Q in the point (x,y,t).

We break up the bound of Q .Denote

$$\sigma_0 = \{(x,y,t) \in \partial G \times (0,T) : \alpha^k \nu_k = 0\},$$

$$\sigma_1 = \{(x,y,t) \in \partial G \times (0,T) : \alpha^k \nu_k > 0\},$$

$$\sigma_2 = \{(x,y,t) \in \partial G \times (0,T) : \alpha^k \nu_k < 0\},$$

Consider in Q the boundary value problem

$$L_0 lu + L_1 u + Mu = f(x,y,t),$$

$$u|_{\partial Q} = 0, \quad \alpha^k u_{x_k}|_{\sigma_2} = 0. \qquad (3)$$

Define the operator d:

$$du = (b^{ij} + \alpha^k a^{ij}_{x_k} - \alpha_0 a^{ij} + a^{ij}_t)u_{x_i x_j} + (b^i + \alpha_0 a^i - \alpha^i a^k_{x_k} + \alpha^i a_0 - a^i_t)u_{x_i} +$$

$$+(a_{0_t} - \alpha_0 a_0)u \equiv d^{ij}u_{x_i x_j} + d^i u_{x_i} + du.$$

Assume that the condition

$$d^{ij} = d^{ji}, \quad \gamma_0 |\xi|^2 \le d^{ij}\xi_i\xi_j \le \gamma_1 |\xi|^2, \quad \forall (x,y,t) \in Q \cup \partial Q, \quad \forall \xi \in \mathbb{R}^{n+m+1} \qquad (4)$$

holds.

Let

$$Q_\tau = Q \cap \{(x,y,t) : 0 < y_1 < \tau\}, \qquad \partial G_\tau = \partial G \cap \{y : 0 < y_1 < \tau\},$$

$$\sigma_{0,\tau} = \{(x.y.t) \in \partial G_\tau \times (0,T) : \alpha^k \nu_k = 0\},$$

$$\sigma_{1,\tau} = \{(x,y,t) \in \partial G_\tau \times (0,T) : \alpha^k \nu_k > 0\},$$

$$\sigma_{2,\tau} = \{(x,y,t) \in \partial G_\tau \times (0,T) : \alpha^k \nu_k < 0\}.$$

For some $h > 0$, define

$$\sigma_{2,h,\tau}=\{(x,y,t)\in\sigma_{2,\tau}:\rho((x,y,t),\partial\sigma_{2,\tau})>h\},\quad \sigma_{2,\tau}^{h}=\sigma_{2,\tau}\ ,\quad \sigma_{2,h,\tau}.$$

Let $E(Q_\tau)$ be a set of functions $\upsilon\in C^2\left(\bar{Q}_\tau\right)$ such that $\upsilon=0$ in $\partial G_\tau\times(0,T)$ and $\alpha^k\upsilon_{x_k}=0$ on $\sigma_{0,\tau}\cup\sigma_{1,\tau}\cup\sigma_{2,\tau}^{h}$ for some $h>0$.

We denote as $H(Q_\tau)$ the Hilbert space obtained by closing $E(Q_\tau)$ with respect to the norm

$$\| u\|_{H(Q_\tau)}=\left\{\int_{Q_\tau}\left(d_1^{ij}u_{x_i}u_{x_j}+u_{y_p}u_{y_q}+u_t^2+u^2\right)dxdydt-\int_{\sigma_{2,\tau}}\alpha^k\nu_k a^{ij}u_{x_i}u_{x_j}ds\right\}^{\frac{1}{2}},$$

where

$$d_1^{ij}=-\frac{1}{2}\alpha^j a_{x_j}^{ij}-\frac{1}{2}a_t^{ij}+\alpha^j a^i+d^{ij}-\frac{1}{2\lambda_0}a^{ij},$$

$$d_1^{ij}=d_1^{ji},\quad \beta_0|\xi|^2\le d_1^{ij}\xi_i\xi_j\le\beta_1|\xi|^2,\quad \forall(x,y,t)\in Q\cup\partial Q,\quad \forall\xi\in\mathbb{R}^{n+m+1}.$$

Now consider bilinear form

$$a(u,\upsilon)=\int_{Q_\tau}\Big[\alpha^k a^{ij}u_{x_i}\upsilon_{x_jx_k}+a^{ij}u_{x_i}\upsilon_{x_jt}+\left(\alpha^k a_{x_j}^{ij}-\alpha^i a^k\right)u_{x_i}\upsilon_{x_j}+$$
$$d^{ij}u_{x_i}\upsilon_{x_j}+\left(d^i-d_{x_j}^{ij}\right)u\upsilon_{x_i}+\left(a_{x_i}^{ij}+a^i+\alpha^i\right)u_{x_i}\upsilon_t+c^{pq}u_{y_p}\upsilon_{y_q}+\left(c^p-c_{y_q}^{pq}\right)u\upsilon_{y_p}+$$
$$u_t\upsilon_t+(\alpha_0+a_0)u\upsilon_t+\left(c_{y_p}^p-c_0-c_{y_py_q}^{pq}+d+d_{x_i}^i+d_{x_ix_j}^{ij}\right)u\upsilon dxdydt.$$

Definition. If $u(x,y,t)\in H(Q_\tau)$ for any $\tau<\infty$ and

$$a(u,\upsilon)=\int_{Q_\tau}f\upsilon dxdydt \tag{5}$$

for an arbitrary function $\upsilon\in E(Q_\tau)$, $\upsilon|_{S_\tau}=0$ where $S_\tau=Q\cap\{(x,y,t):y_1=\tau\}$, then the function $u(x,y,t)$ is said to be a generalized solution of the problem (1), (3) in the domain Q .

2. Energy inequalities

Theorem 1. (Analog of the Saint-Venant principle) Let $-1\le a_{x_i}^{ij}+a^i+a_0\le 0;$ $\theta\equiv d_0-\frac{1}{2}d_{x_ix_j}^{ij}+\frac{1}{2}d_{x_i}^{i}-\frac{1}{2}c_{y_py_q}^{pq}+\frac{1}{2}c_{y_p}^{p}-c_0\le\theta_0<0,\quad \forall(x,y,t)\in Q\cup\partial Q.$ If $u(x,y,t)$ is generalized solution of the problem (1), (3) and $f(x,y,t)=0$ at $y_1\le\tau_2$, then for any τ_1 such that $0\le\tau_1\le\tau_2$, takes place

$$\int_{Q_{\tau_1}}E(u)dxdydt\le\Phi^{-1}(\tau_1,\tau_2)\int_{Q_{\tau_2}}E(u)dxdydt \tag{6}$$

where $E(u)=d^{ij}u_{x_i}u_{x_j}+c^{pq}u_{y_p}u_{y_q}+u_t^2-\theta u^2.$

Here $\Phi(\tau,\tau_2)$ is a solution of the problem

$$\Phi'=-\mu(\tau)\Phi,\quad \tau_1\le\tau\le\tau_2, \tag{7}$$
$$\Phi(\tau_2,\tau_2)=1,$$

$\mu(\tau)$ is an arbitrary continuous function such that

$$0 < \mu(\tau) \le \inf_N \left\{ \int_{S_\tau} E(\upsilon)dxdy'dt \left| \int_{S_\tau} P(\upsilon)dxdy'dt \right|^{-1} \right\}, \tag{8}$$

$$y' = (y_2, y_3, \ldots, y_m),$$

$$P(\upsilon) = -c^{p1}\upsilon\upsilon_{y_p} + \frac{1}{2}\left(c^1 - c^{1q}_{y_q}\right)\upsilon^2, \tag{9}$$

N is the set of continuously differentiable functions in the neighborhood of $\overline{S_\tau}$ which are equal to zero in $\overline{S_\tau} \cap \left(\partial G_\tau \times (0,T)\right)$.

Proof. Assume in (5) $\upsilon = u_m(\Psi(y_1) - 1)$ where $\Psi(y_1) = \Phi(\tau_1, \tau_2)$ if $0 \le y_1 \le \tau_1$, $\Psi(y_1) = \Phi(y_1, \tau_2)$ if $\tau_1 \le y_1 \le \tau_2$, and $\Psi(y_1) = 1$ if $\tau_2 \le y_1$.

$$u_m \in E(Q_\tau), \ \| u_m - u \|_{H(Q_\tau)} \to 0, \quad u \in H(Q).$$

Then

$$a(u - u_m + u_m, u_m(\Psi - 1)) = 0 \ \text{ in } \ Q_{\tau_2}.$$

Therefore

$$a(u_m, u_m(\Psi - 1)) = \delta_m \ \text{ in } \ Q_{\tau_2} \tag{10}$$

where $\delta_m = -a(u - u_m, u_m(\Psi - 1))$.

It is obvious that $\delta_m \to 0$ at $m \to \infty$. Integrating by parts (10), we have

$$\int_{Q_{\tau_2}} E(u_m)(\Psi - 1)dxdydt \le \int_{Q_{\tau_2}} P(u_m)\Psi' dxdydt + \delta_m.$$

Hence

$$\int_{Q_{\tau_2}} E(u_m)(\Psi - 1)dxdydt \le \int_{Q_{\tau_2},\ Q_{\tau_1}} P(u_m)\mu\Psi dxdydt + \delta_m. \tag{11}$$

The estimation (6) follows from (8) and (11) at $m \to \infty$.

Now we will estimate $\mu(y_1)$ in case when S_τ can be included to the $(n+m)$-dimensional parallelepiped which smallest edge is equal to $\lambda(\tau)$. Suppose that

$$\max_{S_\tau}\left\{\left(\frac{1}{2}c^1 - c^{1q}_{y_q}\right), 0\right\} = \gamma(\tau), \quad \max_{S_\tau} c_{p1} = \beta(\tau).$$

Applying the Friedreich and Cauchy-Bunyakovsky inequalities, we have from (9)

$$\left|\int_{S_\tau} P(\upsilon)dxdy'dt\right| \le \left|\int_{S_\tau} c^{p1}\upsilon\upsilon_{y_p}dxdy'dt\right| + \left|\int_{S_\tau} \frac{1}{2}\left(c^1 - c^{1q}_{y_q}\right)\upsilon^2 dxdy'dt\right| \le$$

$$\beta(\tau)\left[\int_{S_\tau} \upsilon^2 dxdy'dt\right]^{\frac{1}{2}}\left[\int_{S_\tau} \upsilon^2_{y_p} dxdy'dt\right]^{\frac{1}{2}} + \gamma(\tau)\int_{S_\tau} \upsilon^2 dxdy'dt \le$$

$$\left(\frac{\beta(\tau)\lambda(\tau)}{\pi\gamma_0} + \frac{\gamma(\tau)\lambda^2(\tau)}{\pi^2\gamma_0}\right)\int_{S_\tau} E(\upsilon)dxdy'dt.$$

Therefore we can set

$$\mu(\tau)=\pi^2\gamma_0\left(\pi\beta(\tau)\lambda(\tau)+\lambda^2(\tau)\gamma(\tau)\right)^{-1}.$$

If $\left(c^1-2c^{1q}_{y_q}\right)\le 0$ in S_τ, then $\gamma(\tau)=0$. Consequently

$$\mu(\tau)=\frac{\pi\gamma_0}{\beta(\tau)\lambda(\tau)}, \qquad (12)$$

Examples. 1. Let at $y_1\ge\tau_1\ge 0$, the domain Q lies inside the rotation body $|y'|\le\frac{M}{2}(y_1+1)$, i.e. $\lambda(y_1)\le M(y_1+1)$, $M>0$. We have from (15)

$$\mu(y_1)=\frac{\pi c(y_1)}{M(y_1+1)}, \qquad c(y_1)=\frac{d_0}{\beta(y_1)}.$$

Suppose that $c(x_1)=c=const>0$.

In this case, from the inequality (6) we have

$$\int\limits_{Q_{\tau_1}} E(u)dxdydt\le\Phi^{-1}(\tau_1,\tau_2)\int\limits_{Q_{\tau_2}} E(u)dxdydt\le\left(\frac{\tau_1+1}{\tau_2+1}\right)^{\pi c}\int\limits_{Q_{\tau_2}} E(u)dxdydt.$$

2. Consider an example of Q for which

$$\lambda(y_1)\le\pi c\left[(y_1+1)^{k-1}\right]^{-1}, \qquad k=const>0.$$

It is clear that if $k>1$, the domain Q is narrowing at $x_1\to\infty$. If $k=1$, then $\lambda(x_1)\le\pi c$ and this case includes domains lying in the band with the width πc. If $0<k<1$, then Q can be extended respectively at $x_1\to\infty$. For this kind of domains, we can assume

$$\mu(y_1)\le(y_1+1)^{k-1}.$$

Then the estimate (6) is valid for considered domains if

$$\Phi^{-1}(\tau_1,\tau_2)=2\exp\left[-(\tau_2+1)^k+(\tau_1+1)^k\right].$$

As a corollary of the Saint-Venant principle, we have the uniqueness theorem for the problem (1), (3) in unlimited domain Q for classes of functions increasing in infinity depending from $\lambda(\tau)$.

Theorem 2. Let $f(x,y,t)=0$ in Q and conditions of theorem 1 hold. If $u(x,y,t)$ is a generalized solution of the problem (1), (3) in Q and for a sequence $\tau_m\to\infty$ at $m\to\infty$ and some $r_*=const>0$,

$$\int\limits_{Q_{\tau_m}} E(u)dxdydt\le\varepsilon(\tau_m)\Phi(r_*,\tau_m) \qquad (13)$$

where $\varepsilon(\tau_m)\to 0$ at $\tau_m\to\infty$, then $u=0$ in Q_{r_*}.

Proof. We have from (6) considering (13)

$$\int\limits_{Q_{r_*}} E(u)dxdydt\le\Phi^{-1}(r_*,\tau_m)\int\limits_{Q_{\tau_2}} E(u)dxdydt\le\varepsilon(\tau_m)\to 0$$

at $\tau_m\to\infty$. Hence $u=0$ in Ω_{d_*}.

Further for any fixed $r_1 > r_*$, we have

$$\Phi(r_*,\tau_m) = e^{\int_{r_*}^{\tau_m} \mu(s)ds} = e^{\int_{r_1}^{\tau_m} \mu(s)ds} e^{\int_{r_*}^{r_1} \mu(s)ds} = c\Phi(r_1,\tau_m).$$

Therefore

$$\int_{Q_{e_1}} E(u)dxdydt \le \Phi^{-1}(r_1,\tau_m) \int_{Q_{\tau_m}} E(u)dxdydt \le \Phi^{-1}(r_1,\tau_m)\varepsilon(\tau_m)\Phi(r_*,\tau_m) =$$

$$c^{-1}\varepsilon(\tau_m) \to 0 \text{ at } \tau_m \to \infty.$$

Hence, $u = 0$ in Q_{r_1}. Since r_1 was chosen arbitrary, $u = 0$ in Q.

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