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ON SPECTRUM OF BILATERAL SHIFT OPERATORS

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Abstract: Studies on bilateral shifts have been done with a lot of consideration on norms and adjoints. In this paper, we give a special focus to spectrum of bilateral shifts. We show that if no x_m vanishes, then $\pi(W)=\{b: j(W)\le |b| s(W)\}$. If finitely many x_m vanish, then $\pi(W)=\{0\}\cup\pi(W')$, where W' the right shift with is weights $x_{m+1}, x_{m+2},..$, where x_k is the last zero weight. If finitely many x_m vanish then $\pi(W)=\{b:|b|\le s(W)\}$.

Keywords: Bilateral shift, spectrum, eigenvalues, Hilbert space.

INTRODUCTION

Studies on the properties of bilateral shifts have been done by many mathematicians. Most of the work done by these mathematicians is on the properties such as norms, adjoints. The spectrum of bilateral shifts has been an interesting area research for both finite dimensional and infinite dimensional Hilbert spaces.

Basic concepts

Definition 2.1: A complete inner product space is called a Hilbert space.

Definition 2.2: A left shift operator acts on a one sided infinite sequence of numbers by $S^*:(a_1,a_2,a_3,...)\rightarrow(a_2,a_3,a_4,...)$ and on a two sided sequence by $T:(a_k)_k^{\infty}\rightarrow(a_{k+1})_{-\infty}^{\infty}$.

Definition 2.3: A right shift operator acts on a one sided sequence of numbers by $S:(a_1,a_2,a_3,...)\rightarrow(0,a_1,a_2,...)$ and on a two sided sequence by $T^{-1}:(a_k)_{-\infty}^{\infty}\rightarrow(a_{k-1})_{-\infty}^{\infty}$.

Definition 2.4: A bilateral shift operator is a shift operator which is both left sided and right sided.
Definition 2.5: The spectrum of an operator T denoted as $\sigma(T)$ is defined as
$\sigma(T)=\{\lambda\in C: T-\lambda I is not invertible in B(X)\}$

MAIN RESULTS

Lemma 3.1: Let B be a bilateral weighted shift with weight sequence $\{\beta_m\}_{m=1}^{\infty}$ and let

K=lim $_{m\rightarrow\infty}$ β_m, then

i. $\|B\|$=K.

ii. σ_p(B)=ϕ, σ_p(B*)={$\lambda \in C : |\lambda| \prec$K}.

iii. σ_r(B)={$\lambda \in$C:$|\lambda| < K$}, σ_c(B)={$\lambda \in C : |\lambda| = K$ },

iv. $\sigma(B) = \sigma(B^*) = \{\lambda \in C : |\lambda| \le K\}$.

v. $\sigma_r(B^*) = \phi$, $\sigma_c(B^*) = \{\lambda \in C : |\lambda| = K\}$,

vi. Γ(B)={$\lambda \in C : |\lambda| < K$}, Π(B)={$\lambda \in C : \lambda = K$}

Proof:

i. We prove that $\|B\| = K$. So σ (B)$\subset$ {$\lambda \in C : |\lambda| \le K$}

ii. For all $\lambda \in C$, we need to prove that N(λI-B)={0}. We suppose that there exists x$\in$H such that (λ I-B)x=0. Then B x=$\sum_{m=1}^{\infty}(\beta x, e_m)e_m = \sum_{m=2}^{\infty}\beta_{m-1}(x, e_{m-1})e_m = \sum_{m=1}^{\infty}\lambda(x, e_m)e_m$. Hence λ(x, e$_1$)=0 and $\beta_{m-1}(x, e_{m-1}) = \lambda(x, e_m)$ for all m≥ 2. Since $\beta_m \ne 0$ for all m≥ 1, so x=0. This is to say that N(λI-B)={0}. Therefore, σ_p(B)=ϕ, we prove that $\sigma_p(B)^* = \{\lambda \in C : |\lambda| < K\}$. Suppose that $0 < |\lambda_0| < K$. We set α_1=1 and $\alpha_{m+1} = \dfrac{\alpha_m \lambda_0}{\beta_m}$ for all m≥ 1.

Since $\lim_{m\to+\infty}\left|\dfrac{\alpha_{m+1}}{\alpha_m}\right| = \lim_{m\to+\infty}\left|\dfrac{\lambda_0}{\beta_m}\right| = \dfrac{|\lambda_0|}{K}$, hence $x_0 = \sum_{m=1}^{\infty}\alpha_m e_m \in H(x_0 \ne 0)$. It follows that $(\lambda_0 I - B^*)x_0 = 0$ and $\lambda_0 \in \sigma_p(B^*)$. We see that $0 \in \sigma_p(B^*)$.So $\{\lambda \in C : |\lambda| < K\} \subset \sigma_P(B^*)$. On the other hand for $|\lambda| = K$, we show that N $(\lambda I - B^*) = \{0\}$. Suppose $x \in H$ and $B^* x = \lambda x$. Then we have

$$\sum_{m=1}^{\infty}(B^* x, e_m)e_m = \sum_{m=1}^{M}(B^* x, e_m)e_m = \sum_{m=1}^{\infty}\lambda(x, e_m)e_m \text{ and}$$

$$\sum_{m=1}^{\infty}|(x, \beta_m e_{m+1})|^2 \le |\lambda|^2\|x\|^2 = K^2\|x\|^2$$

$$For \sum_{m=1}^{\infty}|(x, \beta_m e_{m+1})|^2 \le K^2\sum_{m=1}^{\infty}|(x, e_{m+1})|^2 = K^2(\|x\|^2 - |(x, e_1)|^2,$$

so $(x, e_1) = 0$. From the fact that $(x, \beta_m e_{m+1}) = \lambda(x, e_m)$ and $\beta_m \ne 0 \ \forall m \ge 1$, we have $(x, e_m) = 0 \ \forall m \ge 1$. Hence x=0 and $\sigma_p(B^*) \subset \{\lambda \in C : \lambda \prec K\}$.

iii. For every $\lambda \in C$, we have $\overline{R(\lambda I - B)}$ =(N($\bar{\lambda}I - B^*$))$^{\perp}$. Hence $\overline{R(\lambda I - B)} \ne H$ if and only if $\bar{\lambda} \in \sigma_p(B^*)$. From (ii) above, it follows that $\sigma_r(B) = \sigma_p(B^*) = \{\lambda \in C : \lambda \prec K\}$. Since $\sigma_r(B) \subset \sigma(B) \subset \{\lambda \in C : |\lambda| \le K\}$ and $\sigma(B)$ is a closed set, so $\sigma(B) = \sigma(B^*) = \{\lambda \in C : |\lambda| \le K\}$. For $\sigma_p(B) = \phi$, hence $\sigma_c(B) = \{\lambda \in C : |\lambda| = K\}$.

iv. For every $\lambda \in C$, we have $\overline{R(\lambda I - B^*} = (N(\bar{\lambda}I - B))^{\perp}$=H. It is evident that $\sigma_r(B^*) =$ Γ(B*)=ϕ. From (ii) and (iii), it follows that $\sigma_c(B^*) = \{\lambda \in C : |\lambda| = K$ and $\prod(B^*) = \{\lambda \in C : |\lambda| \le K\}$

v. Since $\sigma_r(B) = \Gamma(B) - \sigma_p(B)$ and $\sigma_p(B) = \phi$, so $\Gamma(B) = \sigma_r(S) = \{\lambda \in C : |\lambda| \prec K\}$. In order to characterise $\pi(B)$, the following concept and notations are needed. For any $T \in B(H)$, $l(T) = \inf\{\|Tx\| : \|x\| = 1\}$ is defined as bound of T. Therefore for every $x \in H$, we have $\|Tx\| \geq l(T)\|x\|$. We have $\lim_{n\to+\infty}[l(T^n)]^{\frac{1}{n}}$ exists. Let $r_1(T) = \lim_{n\to\infty}[l(T^n)]^{\frac{1}{n}}$. Particularly for our operator B, we have that, $L(B^m) = \inf_{k\succ 1}|\beta_k\beta_{k+1}..\beta_{k+m+1}| = |\beta_1\beta_2..\beta_m|$ for all $m \geq 1$

$r_1(B) = \lim_{m\to+\infty}|\beta_1\beta_2...\beta_m|^{\frac{1}{m}} = \lim_{m\to+\infty}\dfrac{|\beta_1| + |\beta_2| + ... + |\beta_m|}{m} = \lim_{m\to+\infty}|\beta_m| = K$ It follows that if $\lambda \in \pi(B)$, then $|\lambda| \geq r_1(B) = K$ since $\delta\sigma(B) \subset \pi(B)$, where $\delta\sigma(B)$ is the boundary of $\sigma(B)$, therefore $\pi(B) = \{\lambda \in C : |\lambda| = K\}$.

Theorem 3.2: Let A be a bilateral weighted shift with weight sequence $\{\alpha_m\}_{m=-\infty}^{\infty}$ and we let $K = \lim_{m\to+\infty}|\alpha_m|$ and $k = \lim_{m\to-\infty}|\alpha_m|$. Then

i. $\|A\| = K$
ii. $\sigma_p(A) = \phi$
iii. $\sigma_p(A^*) = \{\lambda \in C : k < |\lambda| < K\}$
iv. $\sigma_r(A) = \{\lambda \in C : k \prec |\lambda| \prec K\}, \sigma_c(A) = \{\lambda \in C : |\lambda| = korK\}, \sigma(A) = \sigma(A^*) = \{\lambda \in C : k \leq |\lambda| \leq K\}$
v. $\sigma_r(A^*) = \phi, \sigma_c(A^*) = \{\lambda \in C : korK\}, \Gamma(A^*) = \phi, \Pi(A^*) = \{\lambda \in C : k \leq |\lambda| \leq K\}$
vi. $\Gamma(A) = \{\lambda \in C : k < |\lambda| < K\}, \Pi(T) = \{\lambda \in C : |\lambda| = korK\}$

Proof:

i. To prove that $\|A\| = K$, we show that $\sigma(T) \subset \{\lambda \in C : |\lambda| \leq K\}$

ii. For every $\lambda \in C$, we have to prove that $N(\lambda I - A)x = \{0\}$. Now suppose that there exists some $x \in H$ such that $(\lambda I - A)x = 0$. Then

Ax= $\sum_{m=-\infty}^{+\infty}(Ax, e_m)e_m = \sum_{m=-\infty}^{+\infty}\alpha_{m-1}(x, e_{m-1})e_m = \sum_{m=-\infty}^{+\infty}\lambda(x, e_m)e_m$. Hence $\lambda(x, e_m) = \alpha_{m-1}(x, e_{m-1})$ for all $m = 0, \pm1, \pm2, \ldots$ when $\lambda = 0$ since $\alpha_m \neq 0$ for all m,so x=0

Suppose $\lambda \neq 0$ and $|\lambda| \leq K$. From $\lambda(x, e_m) = \alpha_{m-1}(x, e_{m-1})$, we have the following formulas

$$(x, e_m) = \frac{\alpha.\alpha_1..\alpha_{m-1}}{\lambda^m}(x, e_0) \text{ for all } m \geq 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

And $$(x, e_{-m}) = \frac{\lambda^m}{\alpha_{-1}\alpha_{-2}..\alpha_{-m}}(x, e_0) \text{ for all } m \geq 1 \ldots\ldots\ldots\ldots\ldots(2)$$

In (1), we let $b_m = \dfrac{\alpha_0\alpha_1..\alpha_{m-1}}{\lambda^m}$. Then $\lim_{m\to+\infty}\dfrac{|b_{m+1}|}{|b_m|} = \lim_{m\to+\infty}\dfrac{|\alpha_n|}{|\lambda|} = \dfrac{K}{|\lambda|}$. Hence when $|\lambda| \prec K$, $\{|b_m|\}$ is an increasing sequence. So $\lim_{m\to+\infty}|b_m| \succ 0$. Since $\lim_{m\to+\infty}|(x, e_m)| = \lim_{m\to+\infty}|b_m| \bullet |(x, e_0)| = 0$, so $(x, e_0) = 0$. Now from (1) and (2), we have $(x, e_m) = 0$ for all $m = 0, \pm1, \pm2, \ldots$. This means $x = 0$. Suppose $|\lambda| = K$, From

(2) we have $\left|(x,e_0)\right| = \frac{\left|\alpha_{-1}\alpha_{-2}..\alpha_{-n}\right|}{\left|\lambda\right|^m}\left|(x,e_{-m})\right| \le \frac{K^m}{\left|\lambda\right|^m}\left|(x,e_{-m})\right| = \left|(x,e_{-m})\right| = 0$. So $(x,e_0) = 0$. We can see that $x = 0$ from (1) and (2). We have shown that for all $\lambda \in C, N(\lambda I - A) = \{0\}$. Hence $\sigma_p(A) = \phi$.

iii. Suppose $\lambda \in C$ and $x \in H$ such that $A^* x = \lambda x$. Similarly we have $\lambda(x,e_m) = \overline{\alpha}(x,e_{m+1})$ for all $m = 0,\pm1,\pm2,\ldots$ And we have shown that if $|\lambda| \le k$ or $|\lambda| = k$,then $x = 0$. In this case, $\lambda \notin \sigma_p(A^*)$. Suppose $k \prec |\lambda| \prec K$, we choose $p_0 = q_0 = 1$ and we let $p_{n+1} = \frac{p_m \lambda}{\alpha_m}$ for all $m = 0,1,2,\ldots$ and $q_n = \frac{\overline{\alpha} q_{n+1}}{\lambda}$ for all $n = -1,-2,\ldots$. We can show that $x_0 = \sum_{m=-\infty}^{-1} q_m e_m + \sum_{m=0}^{+\infty} p_m e_m \in H (x_0 \ne 0)$.

iv. For every $\lambda \in C$, we have $\overline{R(\lambda I - A^*)} = (N(\overline{\lambda} I - A^*))^{\perp}$. Hence $\sigma_r(A) = \sigma_p(A^*) = \{\lambda \in C : k < |\lambda| < K\}$ directly from (ii) and (iii). Suppose k=0. Since the spectrum of an operator is a compact set, then $\sigma(A) = \sigma(A^*) = \{\lambda \in C : |\lambda| \le K\}$ and $\sigma_c(A) = \{\lambda \in C : |\lambda| = 0 or K\}$. We let k>0 and $|\lambda| < 0$. We turn to prove that $\lambda \in \rho(A)$, where $\rho(A)$ is the regular set of A. Infact for all $x \in H$, we have $\|(\lambda I - A)x\| \ge \|\|Tx\| - |\lambda|\|x\| = \left(\sum_{n=-\infty}^{+\infty} |\alpha_n|^2 \left|(x,e_m)\right|^2\right)^{\frac{1}{2}} - |\lambda|\|x\| \ge (k - |\lambda|\|x\|)$. So $\lambda I - A$ is bounded below by $k - |\lambda|$. Moreover, $R(\lambda I - A) = (N(\overline{\lambda} I - A^*))^{\perp} = H$ follows from (iii). Hence $\lambda \in \rho(A)$. Because of the compactness of the spectrum, we know that $\sigma(A) = \sigma(A^*) = \{\lambda \in C : k \le |\lambda| \le K\}$ and $\sigma_c(A) = \{\lambda \in C : |\lambda| = k or K\}$

v. For every $\lambda \in C$, since $\overline{R(\lambda I - A^*)} = (N(\overline{\lambda} I - A))^{\perp} = H$ or $\sigma_r(A^*) = \Gamma(A^*) = \phi$. From (iii) and (iv), we have $\sigma_c(A^*) = \{\lambda \in C : |\lambda| = k or K\}$ and $\pi(A^*) = \{\lambda \in C : k \le |\lambda| \le K\}$.

vi. Since $\sigma_r(A) = \Gamma(A) - \sigma_P(A)$, hence $\Gamma(A) = \{\lambda \in C : k < |\lambda| < K\}$ follows from (ii) and (iv). Because the weight sequence of A has no zeros, so A is injective. From (ii) it is true that $\pi(A) = \{\lambda \in C : |\lambda| = k or K\}$.

Theorem 3.3: If no x_m vanishes, then $\pi(W) = \{b : j(W) \le |b| s(W)\}$. If finitely many x_m vanish, then $\pi(W) = \{0\} \cup \pi(W')$, where W' is the right shift with weights $x_{m+1}, x_{m+2},..$, where x_k is the last zero weight. If finitely many x_m vanish then $\pi(W) = \{b : |b| \le s(W)\}$.

Proof: If $j(W) = s(W)$, then $\pi(W)$ is contained in and equal to $\{b : |b| = s(W)\}$. Suppose no x_m vanishes and $j(W) < b < s(W)$. Since $\pi(W)$ is closed and has a circular symmetry, then $b \in \pi(W)$. We choose numbers p,q such that $j(W) < p < b < q < s(R)$ and suppose $\varepsilon > 0$. We choose m such that $\left(\frac{b}{q}\right)^m < \varepsilon$ and n such that $\left|x_{n+1} \ldots x_{n+m}\right|^{\frac{1}{m}} > b$. We choose a such that $\left(\frac{p}{b}\right)^a \in \varepsilon$ and c such that $r > m+n$ and $\left|s_{r+1} \ldots s_{r+a}\right|^{\frac{1}{a}} < p$. We define $z = z_i$ by

$z_{n+1} = 1$

$z_s = \dfrac{x_{n+1} \dots x_{s-1}}{b^{s-n-1}}$ if $k + z \le s \le r + a + 1$

$z_s = 0$ If s<n+1 or s>r+a+1

Then $Wz - bz = \sum_{s=n+1}^{r+m+1} \left(\dfrac{x_{n+1} \dots x_s}{b^{s-n-1}} \dfrac{\rho_{s+1} - x_{n+1} \dots x_{s-1}}{b^{s-n}} \rho_s \right)$

And hence

$$\|W_x - bx\|^2 = |x_{r+a+1}|^2 |z_{r+a+1}|^2 + b^2$$

$$\le \|W\|^2 \left(1 + |z_{r+a+1}|^2\right)$$

Also

$\|z\|^2 = \sum |z_i|^2 \ge |z_{n+m+1}|^2 + |z_{r+1}|^2$ .But $|z_{n+r+1}| = {|x_{n+1} \dots x_{n+m}|}/{b^m} > {1}/{\rho}$ and

$\left| {z_{r+a+1}}/{z_{r+1}} \right| = {x_{r+1} \dots x_{r+a}}/{b^m} < \left({p}/{b} \right) < \varepsilon$

So $\dfrac{\|W_z - b_z\|^2}{\|z\|^z} \le \|W\|^2 \dfrac{1 + |Z_{r+a+1}|^2}{|Z_{n+m+1}|^2 + |Z_{r+1}|^2}$

$$\le \|W\|^2 \max \left(\frac{1}{|x_{n+m+1}|^2} \left| \frac{z_{r+a+1}}{z_{r+1}} \right|^2 \right)$$

$$< \varepsilon^2 \|W\|^2$$

And so $b \in \pi(W)$

Now suppose infinitely many x_m vanish and suppose $0 < b < s(R)$. The last assertion of the theorem will follow if we show that necessarily $b \in \pi(W)$. We choose q such that $b < q < s(W)$. Given $\varepsilon > 0$, we choose m such that $(b/q)^m < \varepsilon$ and n such that $(x_{n+1} \dots x_{n+m})^{\frac{1}{m}} > q$. We let s be the first index greater than n+m such that $x_s = 0$. We define $z = (z_i)$ by

$z_{n+1} = 1$

$z_r = \dfrac{x_{n+1} \dots x_{r-1}}{b^{r-n-1}}$ if $n + z \le r \le s$

$z_r = 0$ If $r \le n$ or $r > s$.

Then $Wz - bz = be_{n+1}$, $\|Wz - bz\| = b$ and $\|z\| \ge |z_{n+m+n}| = |x_{n+1} \dots x_{n+m}| / b^m > \left(q/b \right)^m > 1/e$.

So $\|Wz - bz\| / \|z\| < b\varepsilon$ and $b \in \pi(R)$.

If finitely many x_m vanish, then W is the orthogonal sum of W' and a nilpotent operator, and $\pi(W) = \pi(W') \cup \{0\}$, applying the earlier argument of non-zero weights to W, we have the second assertion.

CONCLUSION

Our main focus has been on the study of the spectrum of bilateral shift operator on a finite dimensional Hilbert space. This result also holds for an infinite dimensional Hilbert space.

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