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NON-CONVEX RANDOM DIFFERENTIAL INCLUSION

D. S. Palimkar
Department of Mathematics,
Vasantrao Naik College, Nanded-431603(M.S.) INDIA
E-mail- dspalimkar@rediffmail.com

Abstract: In this paper, I prove the existence of random solution for the first order initial value problem of non-convex random differential inclusion through random fixed point theory.

Keywords: Random differential inclusion, Carathe'odory condition.

AMS Mathematics Subject Classifications: 60H25, 47H10.

STATEMENT OF THE PROBLEM

Let (Ω, A, μ) be a complete σ-finite measure space and let R be the real line. Let $P_p(R)$ denote the class of all non-empty subsets of R with property p. Given a closed and bounded interval $J=[0,T]$ and given a measurable function $q_0:\Omega\to R,$ consider the first order random differential inclusion (in short RDI),

$$\left.\begin{aligned} x'(t,\omega) &\in F\left(t,x(t,\omega),\omega\right) \quad a.e. \quad t\in J \\ x(0,\omega) &= q_0(\omega) \end{aligned}\right\} \tag{1.1}$$

for all $\omega\in\Omega$, where $F: J\times R\times\Omega\to P_p(R)$.

By a random solution of the RDI (1.1) on $J\times\Omega$, means a measurable function $x:\Omega\to AC^1(J,R)$ satisfying for each $\omega\in\Omega,$ $x'(t,\omega)=v(t,\omega)$ for some measurable $v:\Omega\to L^1(J,R)$ such that $v(t,\omega)\in F\left(t,x(t,\omega),\omega\right)$ a.e. $t\in J,$ where $AC^1(J,R)$ is the space of continuous real-valued functions whose first derivative is absolutely continuous on *J*.

When the right hand side multi-valued function is not convex-valued, the geometrical or algebraic multi-valued fixed-point theory is used for proving the existence theorem under certain Lipschitz and monotonicity conditions of multi-valued functions. Here, I will prove the existence result for non-convex case of first order random differential inclusion.

AUXILIARY RESULTS

Let $M(J,R)$ denote the class of real-valued measurable functions on *J* and let $C(J,R)$ denote the space of continuous real-valued functions on *J*. Let $L^1(J,R)$ denote the Banach space of Lebesgue integrable functions on *J* with norm $\|\cdot\|_{L^1}$ defined by $\|x\|_{L^1}=\int_0^T x(t)dt$.

Let $F: J\times R\times R\times\Omega\to P_p(R)$ be a multi-valued mapping. Then for only measurable function $x:\Omega\to C(J,R)$, let

$$S_F(\omega)(x) = \left\{ v \in \mathrm{M}\left(\Omega, M(J,R)\right) \mid v(t,\omega) \in F\left(t, x(t,\omega), \omega\right) \ \ a.e. \ \ t \in J \right\}. \quad (2.1)$$

and

$$S_F^1(\omega)(x) = \left\{ v \in \mathrm{M}\left(\Omega, L^1(J,R)\right) \mid v(t,\omega) \in F\left(t, x(t,\omega), \omega\right) \ \ a.e. \ \ t \in J \right\}. \quad (2.2)$$

This is our set of selection functions for *F* on J × R × Ω.

The integral of the random multi-valued function *F* is defined as

$$\int_0^t F\left(s, x(s,\omega), \omega\right) ds = \left\{ \int_0^t v(s,\omega) ds : v \in S_F^1(\omega)(x) \right\}.$$

Furthermore, if the integral $\int_0^t F\left(s, x(s,\omega), \omega\right) ds$ exists for every measurable function $x : \Omega \to C(J,R)$, then the multi-valued mapping *F* is Lebesgue integrable on *J*.

I need the following definitions in the sequel.

Definition 2.1 A multi-valued mapping $F : J \times R \times \Omega \to \mathrm{P}_{cp}(R)$ is called strong random Carathe'odory if for each $\omega \in \Omega$,

(i) $(t,\omega) \mapsto F(t,x,\omega)$ is jointly measurable for each $x, y \in R$, and

(ii) $x \to F(t,x,\omega)$ is Hausdorff continuous almost everywhere for $t \in J$.

Again, a strong random Carathe'odory multi-valued function *F* is called strong L^1-Carathe'odory if

(iii)For each real number *r* > 0 there exists a measurable function $h_r : \Omega \to L^1(J,R)$ such that for each $\omega \in \Omega$

$$\left\| F(t,x,\omega) \right\|_{\mathrm{P}} = \sup\left\{ |u| : u \in F(t,x,\omega) \right\} \le h_r(t,\omega) \ \text{a. e.} \ \ t \in J$$

for all $x \in R$ with $|x| \le r$.

I quote the following lemmas which are well-known in the literature.

Lemma 2.1 (Lasota and Opial [7]) *Let E be a Banach space. If* $\dim(E) < \infty$ *and* $F : J \times E \times \Omega \to \mathrm{P}_{cp}(E)$ *is strong* L^1*-Carathe'odory, then* $S_F^1(\omega)(x) \neq \emptyset$ *for each* $x \in E$.

Lemma 2.2 (Carathe'odory theroem [5]) *Let E be a Banach space. If* $F : J \times E \to \mathrm{P}_{cp}(E)$ *is strong Carathe'odory, then the multi-valued mapping* $(t,x) \mapsto F\left(t, x(t)\right)$ *is jointly measurable for any measurable E-valued function x on J.*

EXISTENCE RESULT

Let $\mathrm{M}(\Omega, X)$ denote the space of all measurable *X*-valued functions defined on Ω. Define functions $d_{M_1}, d_{M_2} : \mathrm{M}(\Omega, X) \to R^+$ by

$$d_{M_1}(x,y) = \int_0^T \frac{\|x(\omega) - y(\omega)\|}{1 + \|x(\omega) - y(\omega)\|} d\omega$$

and
$$d_{M_2}(x,y) = ess\ \sup\left\{\|x(\omega)\| : \omega \in \Omega\right\}.$$

Then $\mathrm{M}(\Omega, X)$ is a metric space with respect to the above metrics d_{M_1} and d_{M_2}.

Definition 3.1 A multi-valued random operator $Q : \Omega \times X \to \mathrm{P}_{cl}(X)$ is called multi-valued random contraction if there is a measurable function $k : \Omega \to R^+$ such that

$$d_H\left(Q(\omega)x, Q(\omega)y\right) \le k(\omega)\|x - y\|$$

for all $x, y \in X$ and $\omega \in \Omega$, where $0 \le k(\omega) < 1$ on Ω.

I need the following fixed-point theorem for multi-valued random operator is as

Theorem 3.1 (Nowak [8]) *Let* $(\Omega, \mathrm{A}, \mu)$ *be a complete* σ*-finite measure space, X a separable banach space, and let* $Q : \Omega \times X \to \mathrm{P}_{cl}(X)$ *be a random multi-valued contraction. Then* $Q(\omega)$ *has a random fixed point*

I use the following result which come from the classical K. Kuratowskii and C. Ryll-Nardzeuskii selection theorem[6].

Proposition 3.1 *Let* (Ω, A) *be a measurable space and let X be a separable Banach space. Let* $Q : \Omega \to \mathrm{P}_{cl}(X)$ *be a measurable multi-valued operator and let* $\phi : \Omega \to X$ *be a measurable selector. Then for* $\varepsilon > 0$ *there exists a measurable selector* y of $Q(\omega)$ *such that*

$$d\left(\phi(\omega), y(\omega)\right) \le d\left(\phi(\omega), Q(\omega)\right) + \varepsilon, \quad \textit{for all } \omega \in \Omega.$$

I consider the following set of hypotheses for proving the main result.

(A_1) *F* defines a multi-valued mapping $F : J \times R \times \Omega \to \mathrm{P}_{cl}(R)$.

(A_2) *F* is strong random Carathe'odory

(A_3) $(t, \omega) \to F(t, x, \omega)$ is jointly measurable for each $x \in R$.

(A_4) There exists a measurable function $\ell : \Omega \to L^1(J, R)$ satisfying for each $\omega \in \Omega$,

$$d_H\left(F(t,x,\omega), F(t,y,\omega)\right) \le \ell(t,\omega)|x - y| \qquad \text{for all } x, y \in R.$$

(A_5) F is integrally bounded on $J \times R \times \Omega$.

MAIN EXISTENCE RESULT

Theorem 3.4.2 *Assume that the hypotheses* $(A_1)-(A_5)$ *hold. Further, if* $\|\ell(\omega)\|_{L^1}<1$ *for all* $\omega\in\Omega$, *then the RDI (1.1) has a random solution defined on* $J\times\Omega$.

Proof: RDI (1.1) s equivalent to the RII

$$x(t,\omega)\in q_0(\omega)+\int_0^t F\big(s,x(s,\omega),\omega\big)ds,\quad t\in J. \qquad (3.1)$$

Set $X=\mathrm{M}\big(\Omega,C(J,R)\big)$ and define the multi-valued operator $Q:\Omega\times C(J,R)\to \mathrm{P}_p(X)$ by

$$Q(\omega)x(t)=q_0(\omega)+\int_0^t F\big(s,x(s,\omega),\omega\big)ds,\ \ t\in J$$

$$=\big(\mathrm{K}\circ S_F^1(\omega)\big)(x)(t) \qquad (3.2)$$

Where $\mathrm{K}:\mathrm{M}\big(\Omega,L^1(J,R)\big)\to\mathrm{M}\big(\Omega,C^1(J,R)\big)$ is a continuous operator defined by

$$\mathrm{K}v(t,\omega)=q_0(\omega)+\int_0^t v(s,\omega)ds. \qquad (3.3)$$

I show that $Q(\omega)$ is a multi-valued random operator on *X*. First, I show that the multi-valued map $(\omega,x)\mapsto S_F^1(\omega)(x)$ is measurable. Let $f\in\mathrm{M}\big(\Omega,L^1(J,R)\big)$ be arbitrary. Then

$$d\big(f,S_F^1(\omega)(x)\big)=\inf\left\{\|f(\omega)-h(\omega)\|_{L^1}:h\in S_F^1(\omega)(x)\right\}$$

$$=\inf\left\{\int_0^T|f(t,\omega)-h(t,\omega)|dt:h\in S_F(\omega)(x)\right\}$$

$$=\int_0^T\inf\left\{|f(t,\omega)-z|:z\in F\big(t,x(t,\omega),\omega\big)\right\}dt$$

$$=\int_0^T d\big(f(t,\omega),F\big(t,x(t,\omega),\omega\big)\big)dt.$$

But by hypothesis (A_2), the mapping $F\big(t,x(\eta(t),\omega),\omega\big)$ is measurable. Now the function $z\mapsto d\big(z,F(t,x,\omega)\big)$ is continuous and hence the mapping $(t,x,\omega,f)\mapsto d\big(f(t,\omega),F\big(t,x(\eta(t),\omega),\omega\big)\big)$ is measurable from $J\times X\times\Omega\times L^1(J,R)$ into R^+. Now the integral is the limit of the finite sum of measurable functions, and so, $d\big(f,S_F^1(\omega)(x)\big)$ is measurable. As a result, the multi-valued mapping $(\cdot,\cdot)\to S_{F(\cdot)}^1(\cdot)$ is jointly measurable.

Define the multi-valued map ϕ on $J\times X\times\Omega$ by

$$\phi(t,x,\omega)=\left(\mathrm{K}\circ S_F^1(\omega)\right)(x)(t)=\int_0^t F\left(s,x(s,\omega),\omega\right)ds$$

I have shown that $\phi(t,x,\omega)$ is continuous in *t* in the Hausdorff metric on R. Let $\{t_n\}$ be a sequence in *J* converging to $t\in J.$ Then

$$d_H\left(\phi(t_n,x,\omega),\phi(t,x,\omega)\right)$$

$$=d_H\left(\int_0^{t_n} F\left(s,x(s,\omega),\omega\right)ds,\int_0^{t} F\left(s,x(s,\omega),\omega\right)ds\right)$$

$$=d_H\left(\int_J \mathrm{X}_{[0,t_n]}(s)F\left(t,x(s,\omega),\omega\right)ds,\right.$$

$$\left.\int_J \mathrm{X}_{[0,t]}(s)F\left(s,x(s,\omega),\omega\right)ds\right)$$

$$=\int_J \left|\mathrm{X}_{[0,t_n]}(s)-\mathrm{X}_{[0,t]}(s)\right|\left\|F\left(s,x(s,\omega),\omega\right)\right\|_{\mathrm{P}}\ ds$$

$$=\int_J \left|\mathrm{X}_{[0,t_n]}(s)-\mathrm{X}_{[0,t]}(s)\right|\left\|F\left(s,x(s,\omega),\omega\right)\right\|_{\mathrm{P}}\ ds$$

$$=\int_J \left|\mathrm{X}_{[0,t_n]}(s)-\mathrm{X}_{[0,t]}(s)\right|\gamma(s,\omega)y\ \left(\left|x(s,\omega)\right|\right)ds$$

Thus the multi-valued map

$$=\int_J \left|\mathrm{X}_{[0,t_n]}(s)-\mathrm{X}_{[0,t]}(s)\right|\gamma(s,\omega)y\ \left(\left\|x(\omega)\right\|\right)ds$$

$$\to 0 \quad \text{as} \quad n\to\infty.$$

$t\mapsto\phi(t,x,\omega)$ is continuous and hence, by Lemma 2.2, the map $(t,x,\omega)\mapsto\int_0^t F\left(s,x(s,\omega),\omega\right)ds$ is measurable.

Again, since the sum of two measurable multi-valued functions is measurable, the map

$$(t,x,\omega)\mapsto q_0(\omega)+\int_0^t F\left(s,x(s,\omega),\omega\right)ds$$

is measurable. Consequently, $Q(\omega)$ is a random multi-valued operator on X.

I show that $Q(\omega)$ satisfies all the conditions of Theorem 3.1 on X. First, I show that *Q* is closed valued multi-valued random operator on X. Observe that the operator $Q(\omega)$ is equivalent to the composition $K\circ S_F^1(\omega)$ of two operators on $\Omega\times L^1(J,R)$. To show $Q(\omega)$ has closed values, it then suffices to prove that the composition operator $K\circ S_F^1(\omega)$ has closed values on [a, b]. Let $x\in[a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(\omega)(x)$ converging to v in measure. Then, by the definition of $S_F^1(\omega)$, $v_n(t,\omega)\in F\left(t,x(t,\omega),\omega\right)$ a.e. for $t\in J.$ Since $F\left(t,x(t,\omega),\omega\right)$ is closed, $v(t,\omega)\in F\left(t,x(t,\omega),\omega\right)$ a.e. for $t\in J.$ Hence, $v\in S_{F(\omega)}^1(x)$. As a result, $S_F^1(\omega)(x)$ is closed set in $L^1(J,R)$

for each $\omega \in \Omega$. From the continuity of K, it follows that $\left(\mathrm{K} \circ S_F^1(\omega)(x)\right)$ is a closed set in X. Therefore, $Q(\omega)$ is a closed-valued multi-valued operator on [*a*, *b*] for each $\omega \in \Omega$.

Next, I show that $Q(\omega)$ is a multi-valued random contraction on *X*. Let $x, y \in X$ and let $u_1 \in Q(\omega)x.$ Then there is a $v_1 \in S_F^1(\omega)x$ such that

$$u(t,\omega) = q_0(\omega) + \int_0^t v_1(s,\omega)ds$$

for all $t \in J$ and $\omega \in \Omega$. Let $\varepsilon > 0$ be given. Then for above $v_1 \in S_F^1(\omega)x,$ by Proposition 3.1, there is a $v_2 \in S_F^1(\omega)y$ such that

$$\begin{aligned}
|v_1(t,\omega) - v_2(t,\omega)| &\le d\left(v_1(t,\omega), S_F^1(\omega)y(t)\right) + \varepsilon \\
&= d\left(v_1(t,\omega), F\left(t, y(t,\omega), \omega\right)\right) + \varepsilon \\
&= d_H\left(F(t, y(t,\omega),\omega), F\left(t, y(t,\omega),\omega\right) + \varepsilon \\
&\le \ell(t,\omega)|x(t,\omega) - y(t,\omega)| + \varepsilon \\
&\le \ell(t,\omega)\|x(\omega) - y(\omega)\| + \varepsilon
\end{aligned}$$

for all $\omega \in \Omega$. Hence,

$$\begin{aligned}
|u_1(t,\omega) - u_2(t,\omega)| &\le \int_0^t v_1(t,\omega) - v_2(t,\omega)ds + \varepsilon\int_0^t ds \\
&\le \int_0^t \ell(s,\omega)\|x(\omega) - y(\omega)\|ds + \varepsilon\int_0^T ds \\
&\le \|\ell(\omega)\|_{L^1}\|x(\omega) - y(\omega)\| + \varepsilon T
\end{aligned}$$

for all $\omega \in \Omega$. Taking supremum over t, we obtain

$$\|u_1(\omega) - u_2(\omega)\| \le \|\ell(\omega)\|_{L^1}\|x(\omega) - y(\omega)\| + \varepsilon T\,.$$

Since ε is arbitrary, one has

$$\|u_1(\omega) - u_2(\omega)\| \le \|\ell(\omega)\|_{L^1}\|x(\omega) - y(\omega)\|$$

Interchanging the role of u_1 and u_2,

$$d_H\left(Q(\omega)x, Q(\omega)y\right) \le \|\ell(\omega)\|_{L^1}\|x(\omega) - y(\omega)\|$$

for all $x, y \in X$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a multi-valued random contraction on *X* with the contraction constant $\alpha(\omega) = \|\ell(\omega)\|_{L^1} < 1$ for all $\omega \in \Omega$. Hence $Q(\omega)$ has a random fixed point and the set of all random fixed point is closed set in *X*. This further implies that the RDI (1.1) has a random solution and the set of all random solutions is as closed subset of *X* defined on $J \times \Omega$. This completes the proof.

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