Volume 1, No. 1, January 2013

Journal of Global Research in Mathematical Archives

RESEARCH PAPER

Available Online at http://www.jgrma.info

Harvesting, Hopf Bifurcation and Chaos in Three Species Food Chain Model with Beddington–DeAngelis Type Functional Response

Manju Agarwal [1] & Rachana Pathak[2]
Department of Mathematics & Astronomy
Lucknow University, Lucknow-226007, India
[1]E-mail: manjuak@yahoo.com
[2]E-mail: rachanapathak2@gmail.com

Abstract: The dynamical relationship between predator and prey can be represented by the prey functional response which refers to the change in the density of prey attached per unit time per predator as the prey density changes. In this paper, three-species food chain model with Beddington–DeAngelis type functional response is considered and found solution both analytically and numerically. We investigate the Hopf bifurcation and Chaos of the system at mortality rate (a_2) of predator with the help of computer simulations. Butler-Mc Gehee lemma is used to identify the condition which influences the persistence of the system. We also study the effect of Harvesting on prey species. Harvesting has a strong impact on the dynamic evolution of a population. To a certain extent, it can control the long-term stationary density of population efficiently. However, it can also lead to the incorporation of a positive extinction probability and therefore to potential extinction in finite time. Our result suggests that the mortality rate of predator species have the ability to control the chaotic dynamics.

Keywords: Food Chain Model, Stability, Persistence, Harvesting, Chaotic Attractor, Hopf Bifurcation

INTRODUCTION

The traditional mathematical model of predator-prey interactions consists of the following system of two differential equations:

$$\frac{dx}{dt} = P(x) - F(x,y); \qquad x(0) = x_0 \geq 0,$$
$$\frac{dy}{dt} = eF(x,y) - D(y); \qquad y(0) = y_0 \geq 0. \qquad \text{(i)}$$

where x and y represent the prey and predator population sizes respectively, and functions $P(x), D(y)$ describe the intrinsic growth rate of the prey and the mortality rate of the predator, respectively. The function $D(y)$ is assumed to be linear $(D(y) = dy)$, while the function $P(x)$ has a linear $(P(x) = ax)$ or logistic $\left(P(x) = ax\left(1 - \frac{x}{k}\right)\right)$ expression. The function $F(x,y)$ is called "functional response" or "feeding rate" and represents the prey consumption per unit time. The model assumes a linear correspondence between the prey consumption and the predator production through the positive constant e. Among the most popular functional responses used in the modeling of predator-prey systems are the Michaelis-Menten type $F(x,y) = \frac{fxy}{x+c}$ and the ratio-dependent type $F(x,y) = \frac{fxy}{x+by}$. However, in some situations they predict unrealistic population dynamics of the predator or the prey. The Michaelis-Menten type does not account for the mutual competitions among predators [22], while the ratio-dependent type allows unrealistic positive growth rate of the predator at low densities [5, 16, and 23]. The Beddington-DeAngelis functional response $F(x,y) = \frac{fxy}{b+wx+y}$ was introduced independently by Beddington [9] and DeAngelis [2] as a solution of the observed problems in the classical predator-prey theory. It has an extra term in the denominator which models mutual interference between predators and avoids the "low densities problem" of the ratio-dependent type functional response. The Beddington-DeAngelis predator-prey model with a linear intrinsic growth of the prey population, analyzed completely in [3], has the following non dimensional form:

$$\frac{dx}{dt} = x - \frac{axy}{1+x+y}; \qquad x(0) = x_0 \geq 0,$$
$$\frac{dy}{dt} = \frac{exy}{1+x+y} - dy; \qquad y(0) = y_0 \geq 0, \qquad \text{(ii)}$$

where x and y represent the prey and predator population sizes respectively and the positive constants a, e and d represent the generalized feeding rate, generalized conversion efficiency and generalized mortality rate of the predator respectively.

There are numerous studies on the effects of harvesting on population growth. In the context of predator-prey interaction, some studies that treat the populations being harvested as a homogeneous resource include those of Agarwal and Pathak[12,13], Chakraborty et al.[10], .Chaudhuri and Ray [11]. Recently, it is of interest to investigate the possible existence of chaos in biological population. The subjects of chaos and chaos control are growing rapidly in many different fields such biological systems, structural engineering, ecological models, aerospace science, and economics [1, 23]. Food chain modeling provides challenges in the fields of both theoretical ecology and applied mathematics. Determining the equilibrium states and bifurcations of equilibria in a nonlinear system is also an important problem in mathematical models. Gakkhar and Naji [23] investigated a three species ratio dependent food chain (Holling- Tanner Type) model, they also found that there is 'tea-cup' attractor in the system. F. Wang and G. Pang [5] studied a model of a hybrid ratio dependent three species food chain model and they also found chaotic attractor in the system. R.K. Upadhyay [17] studied why chaotic dynamics is rarely observed in natural populations. Many paper [22] studied a predator-prey model with the Michaelis–Menten functional response. Naji and Balasim [18] studied dynamical behavior of a three species food chain model with Beddington-DeAngelis function response and investigated bifurcation and chaotic behavior at conversion rate of prey from predator. Keeping this in mind here we considered Beddington-DeAngelis response in our model and investigate the Hopf bifurcation and Chaos of the predator-prey system at mortality rate of predator numerically. To the best of our knowledge, there are few literatures which have considered the mortality rate of predator, but it has the ability to regulate the population dynamics significantly.

This paper is organized as follows. We start in Section 2 by defining the three species population which consists of prey, predator and top predator. The nonlinear system of differential equations governed this system is introduced. In Section 3, we discuss the equilibrium states and their stability analysis of the three species predator-prey system in Section 4. In Section 5, we derive the sufficient condition for persistence and we investigate the Hopf bifurcation and Chaos of the predator-prey system numerically in Section 6. The main conclusions of the paper are summarized in section 7.

THE FOOD CHAIN MATHEMATICAL MODEL

In this section, we describe the three-species food chain model with Beddington-DeAngelis type functional response with prey, predator and top predator. Such system can be described by the following set of nonlinear differential equations:

$$\frac{dx}{dt} = a_1 x - b_1 x^2 - \frac{wxy}{x+Dy+a_0} - qEx; \qquad x(0) > 0,$$
$$\frac{dy}{dt} = -a_2 y + \frac{w_1 xy}{x+Dy+a_0} - \frac{w_2 yz}{y+D_1 z+b_0}; \qquad y(0) > 0, \qquad (2.1)$$
$$\frac{dz}{dt} = -cz + \frac{w_3 yz}{y+D_1 z+b_0}; \qquad z(0) > 0.$$

where x, y and z are prey, predator and top predator population respectively. a_1, a_2, b_1, $w, w_1, w_2, w_3, c, D, D_1, a_0, q, E$ and b_0 are positive constants. a_1 is the per capita rate of self reproduction for the prey. The parameter a_2 measures how fast the predator y will die when there is no prey to capture, kill and eat. b_1 measures the intensity of competition among individuals of species x for space, food etc. a_0 measures protection provided to prey by its environment, b_0 measures protection provided to predator by its environment, D represents intensity of interference between individuals of the specialist predator and D_1 represents intensity of interference between individuals of the top predator, w_1 measures the efficiency of biomass conversion from prey to predator, w is the per capita rate of predation of the predator, w_2 is the per capita rate of predation of the top predator, w_3 measures the efficiency of biomass conversion from predator to top predator and c is the death rate of the top predator. Moreover, the catch rate function qEx is based on the catch-per-unit-effort (CPUE) hypothesis. Here, q is the catchability coefficient of the predator species and E is the harvesting effort.

To analyze the model (2.1), we need the bounds of dependent variables involved. For this we find the region of attraction in the following lemma.

Lemma 2.1: The set

$$\Omega = \left\{ (x, y, z) : 0 \le x + y + z \le \frac{2(a_1 - qE)^2}{b_1 \alpha} \right\},$$

Where $\alpha = \min\{a_1 - qE, a_2, c\}$

is the region of attraction for all solutions initiating in the interior of the positive octant.

Proof: Let (x, y, z) be any solution with positive initial conditions (x_0, y_0, z_0).

From first equation of model (2.1), we get

$$\frac{dx}{dt} \le (a_1 - qE)x - b_1 x^2 .$$

By usual comparison principle, we have

$$\lim_{t\to\infty} \sup\ x(t) \le \frac{a_1 - qE}{b_1} \quad \text{for all } t \ge 0 ,$$

$$x_{\max} = \frac{a_1 - qE}{b_1}.$$

Now define a function $W(t) = x(t) + y(t) + z(t)$.

Computing the time derivative of $W(t)$ along solutions of system (2.1), we get

$$\begin{aligned} \frac{dW}{dt} &= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}, \\ &\le (a_1 - qE)x - a_2 y - cz, \\ &\le \frac{2(a_1 - qE)^2}{b_1} - \alpha W, \end{aligned}$$

where $\alpha = \min\{a_1 - qE, a_2, c\}$.

Applying a theorem in differential inequalities, we obtain

$$0 \le W(x, y, z) \le \frac{2(a_1 - qE)^2}{b_1 \alpha} + \frac{W(x_0, y_0, z_0)}{\exp(\alpha t)} ,$$

$t \to \infty,$ and for $0 \le W \le \frac{2(a_1 - qE)^2}{b_1\alpha}$. Therefore all solutions of system (2.1) enter into the region

$$\Omega = \left\{(x, y, z) : 0 \le x + y + z \le \frac{2(a_1 - qE)^2}{b_1\alpha}\right\}.$$

This completes the proof of lemma.

EXISTENCE OF EQUILIBRIA

Equating the derivatives on the left hand sides to zero and solving the resulting algebraic equations we find four possible equilibria $E_1(0,0,0), E_2\left(\frac{a_1 - qE}{b_1}, 0.0\right), E_3(\bar{x}, \bar{y}, 0)$ and $E_4(\hat{x}, \hat{y}, \hat{z})$. The trivial equilibrium point $E_1(0,0,0)$ always exists. The equilibrium point $E_2\left(\frac{a_1 - qE}{b_1}, 0.0\right)$ exists on the boundary of the first octant. We show the existence of other equilibria as follows

Existence of $E_3(\bar{x}, \bar{y}, 0)$

Here $\bar{x}, \bar{y}$ are the positive solutions of the following algebraic equations

$$a_1 - b_1\bar{x} - \frac{w\bar{y}}{(\bar{x} + D\bar{y} + a_0)} = 0, \tag{3.1}$$

$$-a_2 + \frac{w_1\bar{x}}{(\bar{x} + D\bar{y} + a_0)} = 0. \tag{3.2}$$

From equation (3.2), we get

$$\bar{y} = \frac{(w_1 - a_2)\bar{x} - a_0 a_2}{D a_2}.$$

Putting the value of $\bar{y}$ in equation (3.1), we get

$$D b_1 w_1 \bar{x}^2 + \{w(w_1 - a_2) - D(a_1 - qE)w_1\}\bar{x} - a_0 a_2 w = 0. \tag{3.3}$$

Equation (3.3) has unique positive solution $x = \bar{x}$, if the following inequalities hold

$$w(w_1 - a_2) > D(a_1 - qE)w_1,\ w_1 > a_2,\ a_1 > qE. \tag{3.4}$$

And for $\bar{y}$ to be positive, we must have

$$\bar{x} > \frac{a_0 a_2}{w_1 - a_2}.$$

Hence the equilibrium $E_3(\bar{x}, \bar{y}, 0)$ exists under the above conditions.

Existence of $E_4(\hat{x}, \hat{y}, \hat{z})$

Here $\hat{x}, \hat{y}$ and $\hat{z}$ are the positive solutions of the system of algebraic equations given below

$$a_1 - b_1\hat{x} - \frac{w\hat{y}}{(\hat{x} + D\hat{y} + a_0)} - qE = 0, \tag{3.5}$$

$$-a_2 + \frac{w_1\hat{x}}{(\hat{x} + D\hat{y} + a_0)} - \frac{w_2\hat{z}}{(\hat{y} + D_1\hat{z} + b_0)} = 0, \tag{3.6}$$

$$-c + \frac{w_3\hat{y}}{(\hat{y} + D_1\hat{z} + b_0)} = 0. \tag{3.7}$$

Straightforward computations show that

$$\hat{y} = \frac{b_1\hat{x}^2 + a_0b_1\hat{x} - (\hat{x} + a_0)(a_1 - qE)}{(a_1 - qE)D - w - b_1D\hat{x}},$$

$$\hat{z} = -\frac{b_0}{D_1} + \frac{(w_3 - c)\left(b_1\hat{x}^2 + a_0b_1\hat{x} - (\hat{x} + a_0)(a_1 - qE)\right)}{cD_1\left((a_1 - qE)D - w - b_1D\hat{x}\right)}.$$

Putting the value of $\hat{y}$ and $\hat{z}$ in equation (3.6), we get

$$B_1\hat{x}^4 + B_2\hat{x}^3 + B_3\hat{x}^2 + B_4\hat{x} + B_5 = 0, \tag{3.8}$$

where

$B_1 = D_1Db_1^2w_3w_1$, $B_2 = Db_1w_3(ww_1 - a_2w - w_1(a_1 - qE)D) - Db_1w_1(w_3D_1a_0b_1 + D_1(a_1 - qE)w_3) + ww_2(w_3 - c)b_1$,

$$B_3 = -wa_0a_2D_1b_1w_3 - (ww_1 - a_2w - w_1(a_1 - qE)D)(w_3D_1a_0b_1 + D_1(a_1 - qE)w_3) - b_1Dw_1w_3a_0(a_1 - qE)D_1 + ww_2a_0b_1(w_3 - c) + w_2w\{b_0b_1Dc + w_3a_0b_1 - ca_0b_1 - (w_3 - c)(a_1 - qE)\},$$

$$B_4 = ww_2a_0\{b_0b_1Dc + w_3b_1a_0 - ca_0b_1 - (w_3 - c)(a_1 - qE)\} + w_2w^2b_0c - b_0(a_1 - qE)Dcw_2w - w_3w_2wa_0(a_1 - qE) + ca_0(a_1 - qE)w_2w,$$

$$B_5 = w^2w_2a_0b_0c - ww_2a_0b_0(a_1 - qE)Dc - ww_2w_3a_0^2(a_1 - qE) + ca_0^2(a_1 - qE)ww_2.$$

Let $F(\hat{x}) = B_1\hat{x}^4 + B_2\hat{x}^3 + B_3\hat{x}^2 + B_4\hat{x} + B_5$,

$$F(0) = ww_2a_0c(wb_0 + (a_1 - qE)a_0) - ww_2a_0(a_1 - qE)(b_0cD + w_3a_0) < 0,$$

$$F\left(\frac{a_1 - qE}{b_1}\right) = B_1\left(\frac{a_1 - qE}{b_1}\right)^4 + B_2\left(\frac{a_1 - qE}{b_1}\right)^3 + B_3\left(\frac{a_1 - qE}{b_1}\right)^2 + B_4\left(\frac{a_1 - qE}{b_1}\right) + B_5 > 0.$$

Now, the sufficient condition for the uniqueness of E_4 is $F'(x) > 0$ and $F'(x)$ is defined at equilibrium point E_4 as

$$F'\left(\frac{a_1 - qE}{b_1}\right) = 4B_1\left(\frac{a_1 - qE}{b_1}\right)^3 + 3B_2\left(\frac{a_1 - qE}{b_1}\right)^2 + 2B_3\left(\frac{a_1 - qE}{b_1}\right) + B_4 > 0.$$

Hence the equilibrium $E_4(\hat{x}, \hat{y}, \hat{z})$ exists under the above conditions.

4. Stability Analysis

Now, in order to investigate the local behavior of model system (2.1) around each of the equilibrium points, the variational matrix V of the point (x, y, z) is computed as,

$$\begin{pmatrix} a_1 - qE - 2b_1x - \frac{wy(Dy+a_0)}{(x+Dy+a_0)^2} & -\frac{(wx^2+a_0wx)}{(x+Dy+a_0)^2} & 0 \\ \frac{w_1y(Dy+a_0)}{(x+Dy+a_0)^2} & -a_2 + \frac{w_1x(x+a_0)}{(x+Dy+a_0)^2} - \frac{w_2z(D_1z+b_0)}{(y+D_1z+b_0)^2} & -\frac{w_2y(y+b_0)}{(y+D_1z+b_0)^2} \\ 0 & \frac{w_3z(D_1z+b_0)}{(y+D_1z+b_0)^2} & -c + \frac{w_3y(y+b_0)}{(y+D_1z+b_0)^2} \end{pmatrix}$$

Let $V_i;\ i = 1,2,3,4$ denote the variational matrix at $E_i;\ i = 1,2,3,4$ respectively.

Hence

$$V_1 = \begin{pmatrix} a_1 - qE & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -c \end{pmatrix}$$

The characteristic equation of V_1 is

$$(a_1 - qE - \lambda)(a_2 + \lambda)(c + \lambda) = 0.$$

The eigenvalues are $\lambda_1 = a_1 - qE, \lambda_2 = -a_2$ and $\lambda_3 = -c$, so E_1 is always a saddle point.

The variational matrix for E_2 is,

$$V_2 = \begin{pmatrix} -(a_1 - qE) & -\frac{(a_1 - qE)w}{(a_0b_1 + a_1 - qE)} & 0 \\ 0 & -a_2 + \frac{(a_1 - qE)w_1}{(a_0b_1 + a_1 - qE)} & 0 \\ 0 & 0 & -c \end{pmatrix}$$

The three eigenvalues of matrix V_2 are $\lambda_1 = -(a_1 - qE), \lambda_2 = -c$ and $\lambda_3 = -a_2 + \frac{(a_1 - qE)w_1}{a_1 - qE + a_0b_1}$.

From the above variational matrix V_2 it is found that the equilibrium point E_2 is locally asymptotically stable provided $\frac{(a_1 - qE)w_1}{a_1 - qE + a_0b_1} < a_2$ and saddle if $\frac{(a_1 - qE)w_1}{a_1 - qE + a_0b_1} > a_2$.

The variational matrix about another equilibrium point E_3 is

$$V_3 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

where

$$a_{11}=a_1-qE-2b_1\bar{x}-\frac{\left(Dw_1\bar{y}^2+a_0w_1\bar{y}\right)}{\left(\bar{x}+D\bar{y}+a_0\right)^2},a_{12}=-\frac{\left(w\bar{x}^2+a_0w\bar{x}\right)}{\left(\bar{x}+D\bar{y}+a_0\right)^2},a_{21}=\frac{Dw_1\bar{y}^2+a_0w_1\bar{y}}{\left(\bar{x}+D\bar{y}+a_0\right)^2},$$

$$a_{22}=-a_2+\frac{w_1\bar{x}^2+a_0w_1\bar{x}}{\left(\bar{x}+D\bar{y}+a_0\right)^2},a_{23}=-\frac{w_2\bar{y}}{\left(\bar{y}+b_0\right)},a_{33}=-c+\frac{w_3\bar{y}}{\left(\bar{y}+b_0\right)}.$$

The three eigenvalues of matrix V_3 are $-c+\frac{w_3\bar{y}}{\left(\bar{y}+b_0\right)}$ and $\frac{\left(a_{11}+a_{22}\right)\pm\sqrt{\left(a_{11}-a_{22}\right)^2+4a_{12}a_{21}}}{2}$. E_3 is locally unstable in $z-$ direction because $-c+\frac{w_3\bar{y}}{\left(\bar{y}+b_0\right)}$ is always positive and if $a_{11}+a_{22}<0$, E_3 is saddle point in $x-y-z$ plane.

However, for the positive point $E_4\left(\hat{x},\hat{y},\hat{z}\right)$ the variational matrix is

$$V_4=\begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & b_{23} \\ 0 & b_{32} & b_{33} \end{pmatrix}$$

where

$$b_{11}=a_1-qE-2b_1\hat{x}-\frac{\left(Dw_1\hat{y}^2+a_0w_1\hat{y}\right)}{\left(\hat{x}+D\hat{y}+a_0\right)^2},b_{12}=-\frac{\left(w\hat{x}^2+a_0w\hat{x}\right)}{\left(\hat{x}+D\hat{y}+a_0\right)^2},b_{21}=\frac{Dw_1\hat{y}^2+a_0w_1\hat{y}}{\left(\hat{x}+D\hat{y}+a_0\right)^2},$$

$$b_{22}=-a_2+\frac{w_1\hat{x}\left(\hat{x}+a_0\right)}{\left(\hat{x}+D\hat{y}+a_0\right)^2}-\frac{w_2\hat{z}\left(D_1\hat{z}+b_0\right)}{\left(\hat{y}+D_1\hat{z}+b_0\right)^2},b_{23}=-\frac{w_2\hat{y}\left(\hat{y}+b_0\right)}{\left(\hat{y}+D_1\hat{z}+b_0\right)^2},b_{32}=\frac{w_3\hat{z}\left(D_1\hat{z}+b_0\right)}{\left(\hat{y}+D_1\hat{z}+b_0\right)^2},$$

$$b_{33}=-c+\frac{w_3\hat{y}\left(\hat{y}+b_0\right)}{\left(\hat{y}+D_1\hat{z}+b_0\right)^2}.$$

By the Routh-Hurwitz's condition, E_4 is locally asymptotically stable provided the following conditions are satisfied: $A_1>0, A_3>0,$ and $A_1A_2>A_3;$ where $A_i, i=1,2,3$ are the coefficients of the characteristic equation of $V_4=[b_{ij}];i,j=1,2,3.$

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0,$$

with

$$A_1=-b_{11}-b_{22}-b_{33},\ A_2=b_{11}b_{22}+b_{11}b_{33}+b_{22}b_{33}-b_{12}b_{21}-b_{23}b_{32},\ A_3=b_{12}b_{21}b_{33}-b_{11}b_{22}b_{33}+b_{11}b_{23}b_{32}.$$

PERSISTENCE

Biologically, persistence means the survival of all populations in future time. Mathematically, persistence of a system means that strictly positive solutions do not have omega limit points on the boundary of a non-negative cone. A population $x(t)$ is said to be uniformly persistent if there exists an $\delta>0,$ independent of $x(0)>0$ such that $\liminf_{t\to\infty} x(t)>\delta$. We say that a system persists uniformly whenever each component persists uniformly. Stability theory of ordinary differential equations is used to analyze the model.

Theorem (5.1): Let $\frac{(a_1-qE)w_1}{a_1-qE+a_0b_1}>a_2$ holds, then the system (2.1) persists (does not persist) if E_4 exist (does not exist).

Proof: To prove this theorem, we have to show that there are no omega limit points on the axes of orbits initiating in the interior of positive octant. Suppose u is a point in the positive octant and $\theta(u)$ is the orbit through u and ω is the omega limit set of the orbit through u. Note that $\omega(u)$ is bounded.

We claim that E_1 does not belong to $\omega(u)$. If $E_1 \in \omega(u)$, then by Butler McGehee lemma (Butler et al., 1986) there exists a point v in $\omega(u) \cap M^s(E_1)$ where $M^s(E_1)$ denote the stable manifold of E_1. Since $\theta(v)$ lies in $\omega(u)$ and $M^s(E_1)$ in the $y-z$ plane, we conclude that $\theta(v)$ is unbounded, which is a contradiction.

Now, we show that $E_2 \notin \omega(u)$ if $E_2 \in \omega(u)$, the condition $\frac{(a_1 - qE)w_1}{a_1 - qE + a_0 b_1} > a_2$ implies that E_2 is a saddle point and $M^s(E_2)$ is the $x-z$ plane, again we conclude that an unbounded orbit lies in $\omega(u)$, a contradiction.

Next, $E_3 \notin \omega(u)$. If $E_3 \in \omega(u)$, for otherwise, since E_3 is a saddle point which follows from the condition $-c + \frac{w_3 \bar{y}}{(\bar{y} + b_0)} > 0$, by Butler McGehee lemma there exists a point v in $\omega(u) \cap M^s(E_3)$. Now $M^s(E_3)$ is the $x-y$ plane (if $a_{11} + a_{22} < 0$) implies that an unbounded orbit $\theta(v)$ lies in $\omega(u)$, which is a contrary to the boundedness of the system.

Thus, $\omega(u)$ lies in the positive octant and system (2.1) are persistent. Finally, since only the closed orbits and the equilibria form the omega limit set of the solutions on the boundary of R_+^3 and system (2.1) is dissipative, by main theorem in Butler et al. (1986) this implies that system (2.1) is uniformly persistent.

NUMERICAL SIMULATION FOR CHAOS AND BIFURCATION

Numerical integration is used to investigate the global dynamical behavior of the system (2.1). The objective is to show Hopf bifurcation and to explore the possibility of chaotic behavior in system (2.1). If we write the autonomous system (2.1) in the form:

$$\dot{v} = F(v, \mu)$$

where $v = (x, y, z), \mu = (a_0, a_1, a_2, b_0, b_1, w, w_1, w_2, w_3, c, D, D_1, q, E)$

We say that an ordered pair (v_0, μ_0) is a Hopf bifurcation point if,

(i) $F(v_0, \mu_0) = 0$

(ii) $J(v, \mu)$ has two complex conjugate eigenvalues $\lambda_{1,2}$ around (v_0, μ_0), $\lambda_{1,2} = a(v, \mu) \pm Ib(v, \mu)$

(iii) $a(v_0, \mu_0) = 0, b(v_0, \mu_0) \neq 0$

(iv) The third eigenvalue $\lambda_3(v_0, \mu_0) \neq 0$

Extensive numerical simulations are carried out for various values of parameters and for different sets of initial conditions. We take the parameters of the system (2.1) as

$a_1 = 1, b_1 = 1, w = 1.667, D = 0.334, a_0 = 0.334, w_1 = 1.667, w_2 = 0.05, \mathrm{q} = 0.5, \mathrm{E} = 0.5, \mathrm{D}_1 = 0.5,$

$b_0 = 0.6, c = 0.01$ and $w_3 = 0.05.$

We consider the system

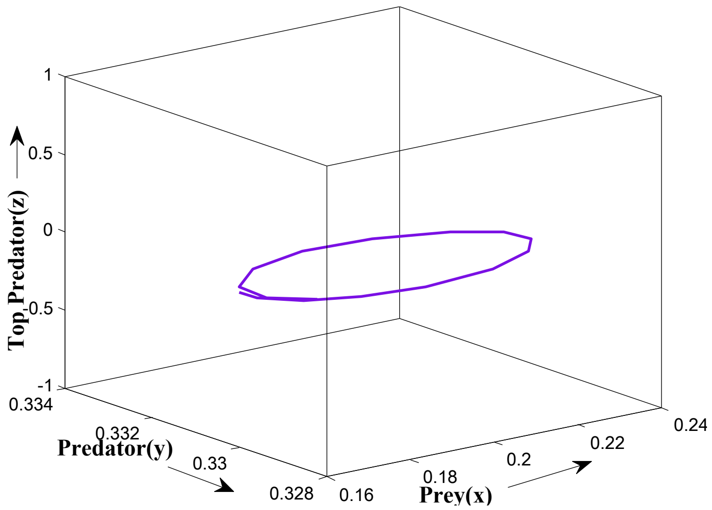
$$\frac{dx}{dt} = 1x - 1x^2 - \frac{1.667xy}{(x + 0.334y + 0.334)} - (0.4)(0.5)x,$$
$$\frac{dy}{dt} = -a_2 y + \frac{1.667xy}{(x + 0.334y + 0.334)} - \frac{0.05yz}{(y + 0.5z + 0.6)}, \tag{6.1}$$
$$\frac{dz}{dt} = -0.01z + \frac{0.05yz}{(y + 0.5z + 0.6)}.$$

The system (6.1) always has nonnegative equilibria $E_1(0,0,0)$ and $E_2(0.75,0,0)$. The system (6.1) has positive equilibria $E_3(\bar{x},\bar{y},0)$ and $E_4(\hat{x},\hat{y},\hat{z})$ if and only if $a_2 \in (0,0.9)$.

Now take $a_2 = 0.2443$, $\mu_1 = (0.334,1,0.2443,0.6,1,1.667,1.667,0.05,0.05,0.01,0.334,0.5,0.5,0.5)$

The coordinates of E_4 and the corresponding eigenvalues are $v_1 = (0.07400681,0.191381,0.331051)$ $\lambda_{1,2} = 0.0000027 \pm 0.357738i$ and $\lambda_3 = -0.00169963$.

In this way ordered pair (v_1, μ_1) is satisfied above all conditions (i-iv). So ordered pair (v_1, μ_1) is Hopf point.



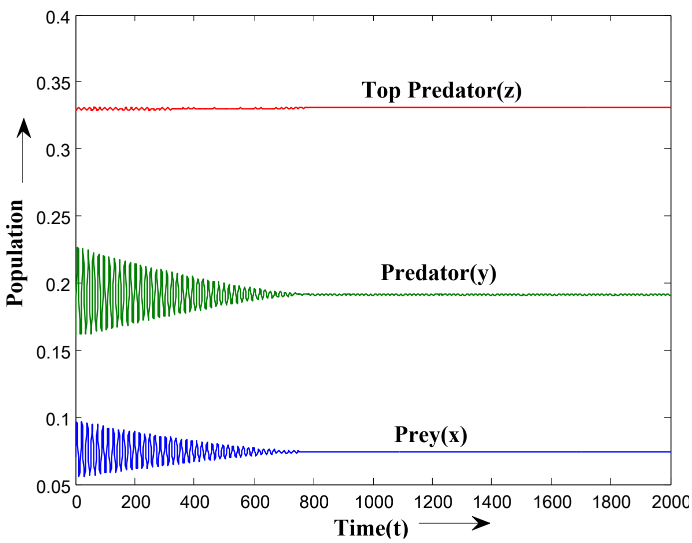


Figure (1)

Here $x(0) = 0.07, y(0) = 0.16, z(0) = 0.33,$ $a_1 = 1, b_1 = 1, w = 1.667, D = 0.334, a_0 = 0.334$
$a_2 = 0.2443, w_1 1.667, w_2 = 0.05, D_1 = 0.5, b_0 = 0.6, c = 0.01, w_3 = 0.05, q = 0.5, E = 0.5$

For $a_2 = 0.2 < 0.2443$, $\mu_1 = (0.334,1,0.2,0.6,1,1.667,1.667,0.05,0.05,0.01,0.334,0.5,0.5,0.5)$

The coordinates of E_4 and the corresponding eigenvalues are $v_1 = (0.0592976,0.189132,0.313058)$, $\lambda_{1,2} = 0.00186926 \pm 0.333809i, \lambda_3 = -0.00162342$. All eigenvalues have not negative real parts, only λ_3 has negative real part, so E_4 is always saddle point at $a_2 = 0.2$.

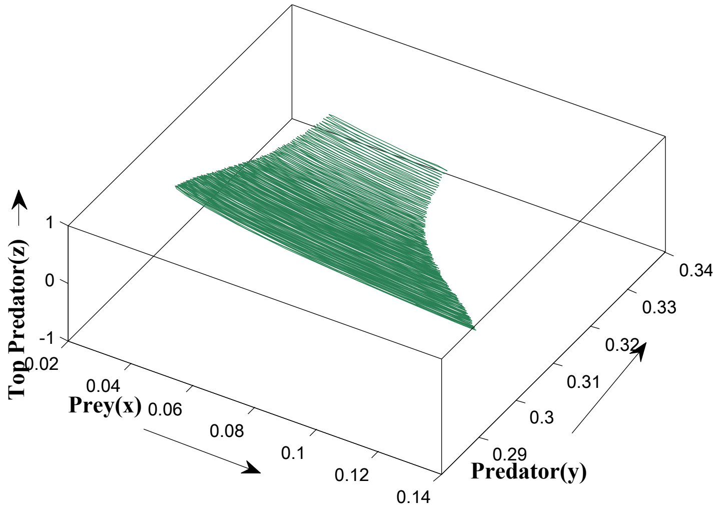


Figure 2(a)

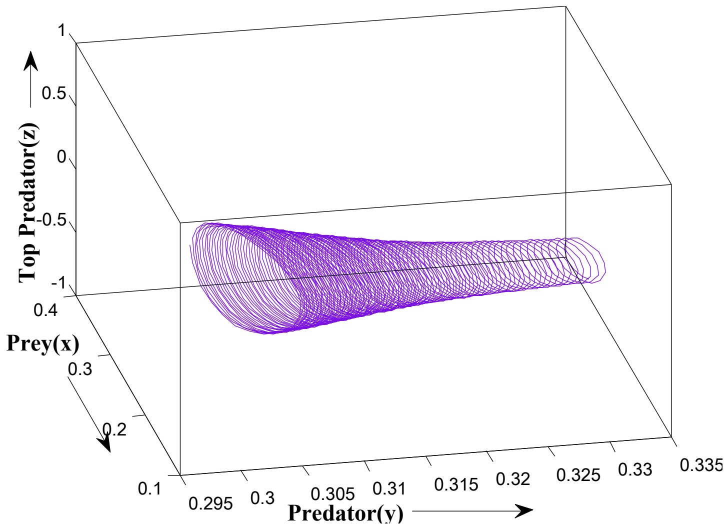


Figure 2(b)

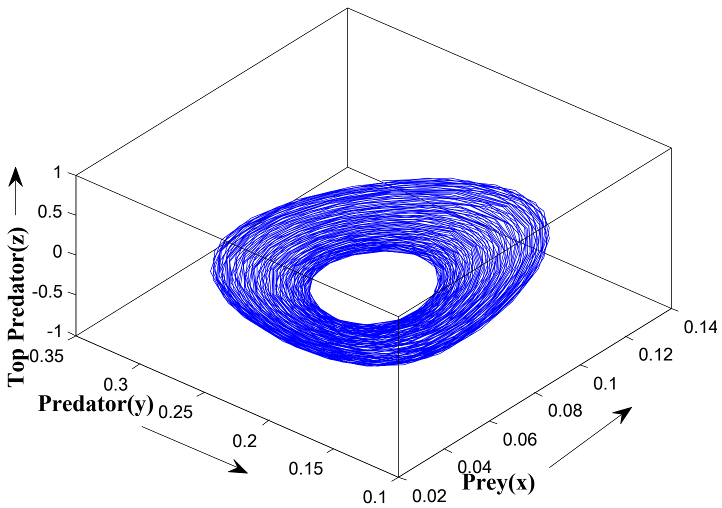


Figure 2 (c)

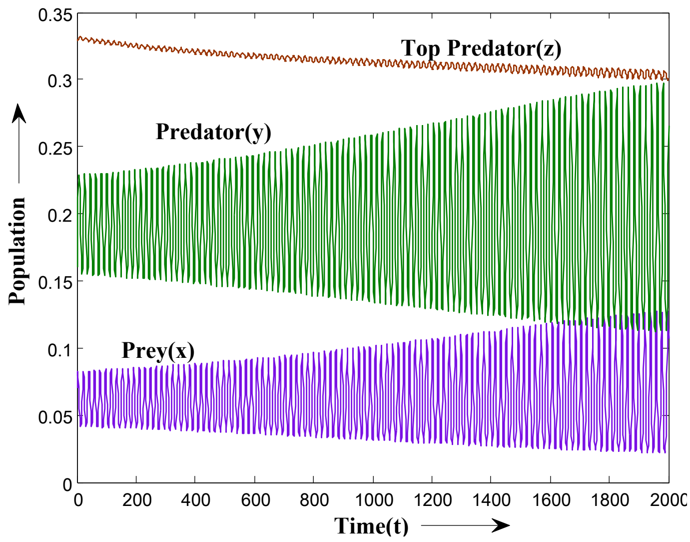


Figure 2 (d)

Figure (2)

Here $x(0) = 0.07, y(0) = 0.16, z(0) = 0.33,$ $a_1 = 1, b_1 = 1, w = 1.667, D = 0.334, a_0 = 0.334$
$a_2 = 0.2, w_1 1.667, w_2 = 0.05, D_1 = 0.5, b_0 = 0.6, c = 0.01, w_3 = 0.05, q = 0.5, E = 0.5$

3D view of the chaotic attractor {Fig 2(1), 2(b), 2(c)}.Sensitive dependence on initial conditions {Fig 2(d)}

But if we take $a_2 = 0.4 > 0.2443$, $\mu_1 = (0.334,1,0.4,0.6,1,1.667,1.667,0.05,0.05,0.01,0.334,0.5,0.5,0.5)$ .The coordinates of E_4 and the corresponding eigenvalues are $v_1 = (0.134339,0.197307,0.378454)$, $\lambda_{1,2} = -0.0137069 \pm 0.410251i, \lambda_3 = -0.0019016$. All eigenvalues have negative real parts, so equilibrium point E_4 is locally asymptotically stable at $a_2 = 0.4$.

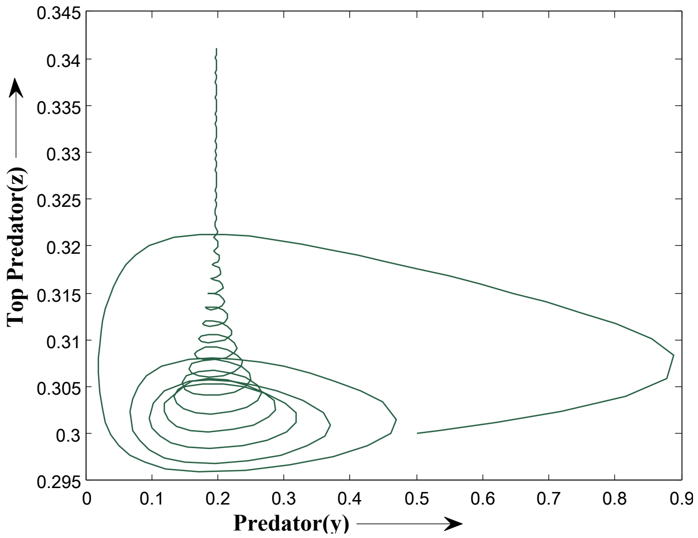


Figure 3(a)

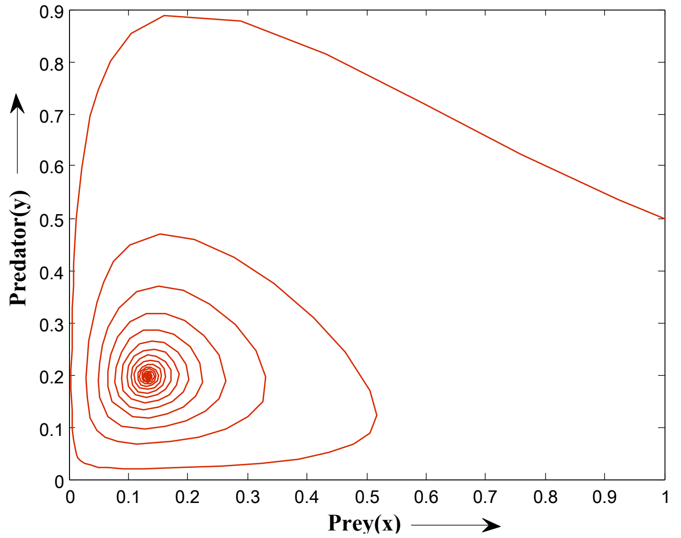


Figure 3(b)

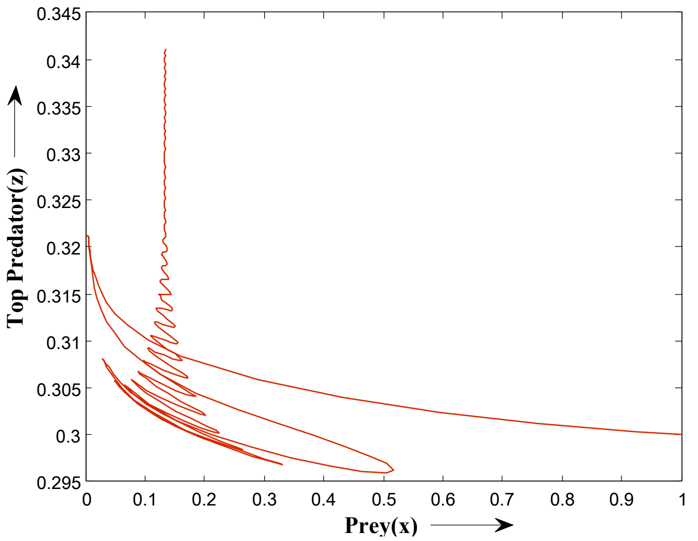


Figure 3(c)

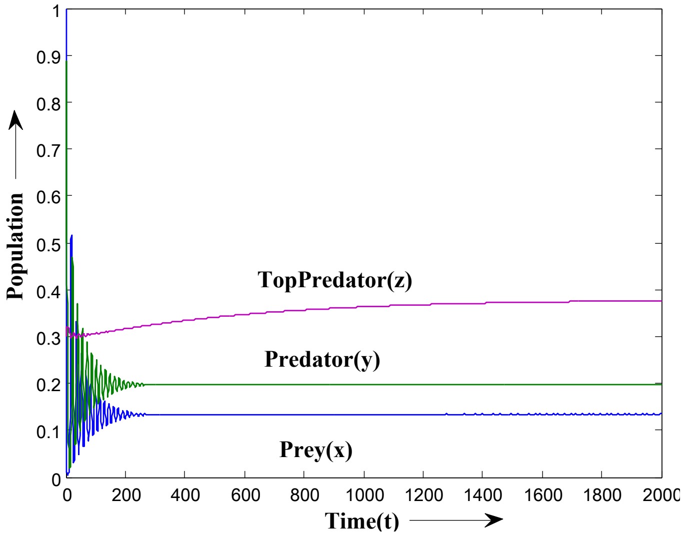


Figure 3 (d)

Figure (3)

Here $x(0)=1, y(0)=0.5, z(0)=0.3,\ a_1=1, b_1=1, w=1.667, D=0.334, a_0=0.334$

$a_2=0.4, w_1 1.667, w_2=0.05, D_1=0.5, b_0=0.6, c=0.01, w_3=0.05, q=0.5, E=0.5$

Phase portrait of the system (6.1) showing that E_4 is locally asymptotically stable {Fig 3(a), 3(b), 3(c)}. x, y, z approach to their equilibrium values in finite time {Fig 3(d)}.

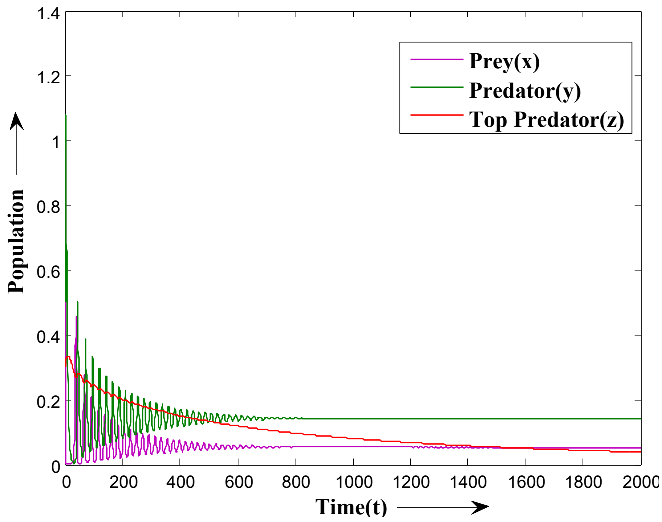


Figure 4(a)

Here $x(0)=1, y(0)=0.5, z(0)=0.3,\ a_1=1, b_1=1, w=1.667, D=0.334, a_0=0.334$

$a_2=0.2, w_1 1.667, w_2=0.05, D_1=0.5, b_0=0.6, c=0.01, w_3=0.05, q=0.8, E=0.5$

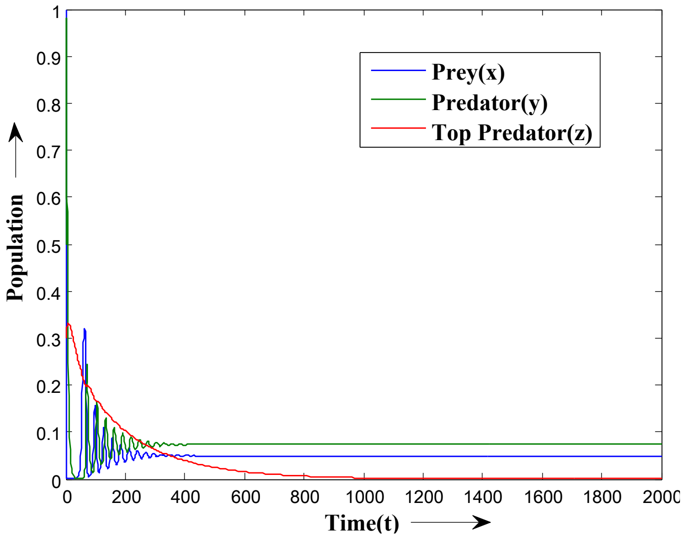


Figure 4(b)

Figure (4)

Here $x(0)=1, y(0)=0.5, z(0)=0.3,\ a_1=1, b_1=1, w=1.667, D=0.334, a_0=0.334$
$a_2=0.2, w_1 1.667, w_2=0.05, D_1=0.5, b_0=0.6, c=0.01, w_3=0.05, q=1.3, E=0.5$

It has been showed numerically that the only Hopf point is found when $a_2=0.2443$ in four digits. E_4 is unstable when $a_2<0.2443$ and stable when $a_2>0.2443$.The numerical study presented here shows that, using the parameter a_2 as control, it is possible to break the stable behavior of the system (6.1) and drive it to an unstable state. From comparing the figure 2(d) and 4(a), we get system become stable at $a_2=0.2$ when we increase the value of q but we can't increase the value of q greater than 1.2 because at 1.3 the population of top predator become extinct, see figure 4(b). So it is possible to keep the population levels at a required state using the above control. A typical chaos just as we observed. The unique character of chaotic dynamics may be seen most clearly by sensitivity to initial conditions. That is, a small change in initial conditions may lead to different dynamic behaviors.

CONCLUSION

We have considered the dynamic behavior of a three-species food chain with Beddington –DeAngelis type functional response. We have obtained conditions for the existence of different equilibria and discussed their stabilities in local manner by using stability theory of differential equations. The system persists under condition derived by Butler McGehee lemma. We have also observed in the numerical simulation that the dynamics of a population may dramatically be affected by small changes in the value of the parameter a_2, at the same time. All our important mathematical findings are numerically verified in section 7.5 and graphical representation of a variety of solutions of the system (2.1) are depicted using MATLAB. Our numerical study shows that, using the parameter a_2 as control, it is possible to break the stable behavior of the system and drive it to an unstable state. Also it is possible to keep the levels of the populations at a stable state using the above control. It is well known that natural populations of plants and animals neither increase indefinitely to blanket the world nor become extinct (except in some rare cases and due to some rare reasons). Hence, in practice, we often want to reduce the predator (y) to an acceptable level in finite time.

REFERENCES

[1.] A. Hasting and T.Powell, Chaos in three species food chain, Ecology, 72(1991), 896-903.

[2.] D. L. DeAngelis, R. A. Goldstein and R. V. O'Neill, A model for trophic interaction, Eocolgy, 56(1975), 881-892.

[3.] D.T.Dimitrov and H.V Kojouharov, Complete mathematical analysis of predator-prey models with linear prey growth and Beddington-DeAngelis functional response, Appl. Math. Comput. 162:2 (2005) 523-538.

[4.] E.Ott, Chaos in dynamical systems, Combridge University Press, Combridge, 1993.

[5.] F.Wang and G.Pang, Chaos and Hopf bifurcation of a hybrid ratio-dependent three species food chain, Chaos Solitons and Fractals, 36(2008), 1366-1376.

[6.] H. Baek, Dynamic complexities of a three-species Beddington-DeAngelis system with impulsive control strategy, Acta Applicandae Mathematicae, 110(1) (2010), 23-38.

[7.] H. K. Baek, Qualitative analysis of Beddington-DeAngelis type impulsive predator-prey models, Nonlinear Analysis: Real World Applications, 11(3) (2010), 1312–1322.

[8.] J. Hale, Theory of functional differential equations, New York: Springer, 1977.

[9.] J. R. Beddington, Mutual interference between parasites or predator and its effect on searching efficiency, J. Animal Ecol., 44(1975), 331-340.

[10.] K. Chakraborty, M. Chakraborty and T. K. Kar, Optimal control of harvest and bifurcation of a prey-predator model with stage structure, Applied Mathematics and Computation, 217(2011),8778-8792.

[11.] K.S. Chaudhuri and S.S. Ray, On the combined harvesting of a prey-predator system, J. Biol. Sys., 4(1996), 373-389.

[12.] M. Agarwal and R. Pathak, Modeling the effect of harvesting of the vegetation biomass and grazer population on predator population with habitat complexity, International Journal of Mathematical Archive, 2(11), 2011, 2119-2134.

[13.] M. Agarwal and R. Pathak, Persistence and optimal harvesting of prey -predator model with Holling Type III functional response, International Journal of Engineering, Science and Technology, 4(3), 2012, 78-96.

[14.] M.E.Gilpin, Spiral chaos in a predator-prey model, American Naturalist, 107(1979), 306-308.

[15.] R.Arditi and H. Saiah, Empirical evidence of the role of predator-prey theory, Ecology 73(1982) 1530-1535.

[16.] Rukiye Kara and Mehmet Can, Hopf point Analysis for ratio-dependent food chain models, Mathematical and Computational Applications, 13(1), 2008, 9-22.

[17.] R.K.Upadhyay and V.Rai, Why chaos is rarely observed in natural populations, Chaos, Solitons and Fractals, 8(12), 1997, 1933-1989.

[18.] R.K.Naji and A.T.Balasim, Dynamical behavior of a three species food chain model with Beddington-DeAngelis functional response, Chaos Solitons and Fractals, 32(2007), 1853-1866.

[19.] R.M.May, Stability and Complexity in Model Ecosystem (Princeton Univ. Press, Princeton 1974).

[20.] R.S.Cantrell and C.Consner, On the dynamics of predator–prey Models with the Beddington–DeAngelis Functional response, J. Math Anal. Appl., 257(2001), 206–22.

[21.] S. Gakkhar and R. K. Naji, Existence of chaos in two-prey, one-predator system, Chaos,Solitions and Fractals , 17(2003), 639-649.

[22.] S.B Hsu, T.W.Hwang and Y. Kuang, Global analysis of the Michaelis-Menten-type ratio dependent predator-prey system, J. Math. Biol. 42 (2001) 489-506.

[23.] S. Gakkar and R. K. Naji, Order and chaos in predator to prey ratio-dependent food chain, Chaos, Solitons and Fractals 18(2003) 229-239.

[24.] W.M.Schaffer, Order and chaos in ecological systems, Ecology, 66(1985), 93-106.

[25.] W.M.Schaffer and M.Kot, Chaos in ecological systems, the coals that Newcastle forgot, Trends in Ecology and Evolution, 1(1986a), 58-63.