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ON A SUBCLASS OF MEROMORPHIC STARLIKE FUNCTIONS WITH ALTERNATING COEFFICIENTS

V.SRINIVAS
Department of Mathematics,
S.R.R. GOVT. DEGREE & P.G. COLLEGE
KARIMNAGAR, DIST.KARIMNAGAR,
TELANGANA-505001., Mobile: 09849432324
Email: vsrinivas079@gmail.com

RAJKMAR N. INGLE
DEPARTMENT OF MATHEMATICS,
BAHIRJI SMARAK MAHAVIDYALAYA,
BASMATHNAGAR,Dist.HINGOLI(M.S.)

P.Thirupathi Reddy
Email:reddypt2@yahoo.com
Department of Mathematics ,
Kakatiya University,
Warangal-506009, TELANGANA.

Abstract: The aim of the present paper is to introduce a new subclass $\sum_{w,a}^{*}(A,B)$ of meromorphic starlike functions with alternating coefficients in *E= { z: 0<| z | < 1}* and investigate coefficients estimates, distortion properties and radius of convexity for the class. Furthermore it is shown that the class $\sum_{w,a}^{*}(A,B)$ is closed under convex linear combinations, convolutions and integral transforms.

Keywords: Meromorphic, Starlike, Convolution and Integral transforms.

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1. INTRODUCTION: Let A be the class of functions f normalized by

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$E=\{z\in\mathbb{C}:|z|<1\}.$$

As usual, we denote by S the subclass of A, consisting of functions which are also univalent in E.We recall here the definitions of the well-known classes of starlike function and convex function :

$$ST=\left\{f\in A:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,z\in E\right\} \tag{1.2}$$

and

$$CV=\left\{f\in A:\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>0,z\in E\right\} \tag{1.3}$$

Let w be a fixed point in E and $A(w)=\{f\in H(E):f(w)=f'(w)-1=0\}$.In [5] , Kanas and Ronning introduced the following classes

S_w={ $f\in A(w)$: f is univalent in E }.

$$ST_w=\left\{f\in A(w):\operatorname{Re}\left(\frac{(z-w)f'(z)}{f(z)}\right)>0,z\in E\right\} \tag{1.4}$$

And

$$CV_w=\left\{f\in A(w):1+\operatorname{Re}\left(\frac{(z-w)f''(z)}{f'(z)}\right)>0,z\in E\right\} \tag{1.5}$$

Later, Acu and Owa [1] studied the class extensively.

The class ST_w is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$, and the corresponding class CV_w is defined by the property that the image of any circular arc centered at w is convex. We observed that the definitions are somewhat similar to the ones introduced by Goodman in [3] and [4] for uniformly starlike and convex functions, except that in this case the point w is fixed .

Let $\sum_w$ denote the subclass of $A(w)$ consisting of the function of the form

$$f(z)=\frac{1}{z-w}+\sum_{n=1}^{\infty}a_n(z-w)^n \tag{1.6}$$

The functions f in $\sum_w$ are said to be starlike functions of order β if and only

$$\operatorname{Re}\left\{\frac{-(z-w)f'(z)}{f(z)}\right\}>\beta\ ,\quad ((z-w)\in E). \tag{1.7}$$

For some β $(0 \le \beta < 1)$. We denote by $ST_w(\beta)$ the class of all starlike functions of order β.Similarly, a function f in $\sum_w$ is said to be convex of order β if and only

$$\operatorname{Re}\left\{-1-\frac{(z-w)f''(z)}{f'(z)}\right\} > \beta, \quad ((z-w) \in E). \tag{1.8}$$

For some β $(0 \le \beta < 1)$. We denote by $CV_w(\beta)$ the class of all convex functions of order β.

Let $\sum_{w,a}$ denote the class of functions of the form

$$f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n (z-w)^n \ , a_n \ge 0 \tag{1.9}$$

that are analytic in E .

2. MAIN RESULTS

Definition 1: Let $\sum_{w,a}^{*}(A,B)$ denote the subclass of consisting of functions *f(z)* which satisfy

$$\left|\frac{(z-w)f'(z)}{f(z)}+1\right| < \left|A + B\frac{(z-w)f'(z)}{f(z)}\right| \tag{1.10}$$

for $-1 \le A < B, 0 < B \le 1$ and $((z-w) \in E)$.

2. COEFFICIENT ESTIMATES

Theorem 1: Let $f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n (z-w)^n \ , a_n \ge 0$ be regular in *E*. Then *f(z)* is in the class $\sum_{w,a}^{*}(A,B)$ if and only if

$$\sum_{n=1}^{\infty} \{(n+1) + (A+Bn)\}\, a_n \le B - A. \tag{2.1}$$

for $-1 \le A < B$ and $0 < B \le 1$.

Proof: Suppose that $f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n (z-w)^n \ , a_n \ge 0$ is in $\sum_{w,a}^{*}(A,B)$ then

$$\left| \frac{\frac{(z-w)f'(z)}{f(z)}+1}{A+B\frac{(z-w)f'(z)}{f(z)}} \right| = \left| \frac{\sum_{n=1}^{\infty}(-1)^{n-1}(n+1)a_n(z-w)^n}{(B-A)\frac{1}{(z-w)}-\sum_{n=1}^{\infty}(-1)^{n-1}(A+Bn)a_n(z-w)^n} \right| < 1$$ for all $(z\text{-}w) \in E$. Since $\mathrm{Re}(z-w) \le |z-w|$ for all $(z\text{-}w)$, we have

$$\mathrm{Re}\left\{ \frac{\sum_{n=1}^{\infty}(n+1)\,a_n(z-w)^n}{(B-A)\frac{1}{(z-w)}-\sum_{n=1}^{\infty}(A+Bn)\,a_n(z-w)^n} \right\} < 1, ((z-w) \in E) \tag{2.2}$$

Now choose the values of $(z\text{-}w)$ on the real axis so that $\frac{(z-w)f'(z)}{f(z)}$ is real. Upon clearing the denominator in (2.2)

we obtain $$\sum_{n=1}^{\infty}\{(n+1)+(A+Bn)\}a_n \le B-A$$

Conversely, suppose that (2.1) holds for all admissible values of A and B.

We have

$$H(f,f') = |(z-w)f'(z)+f(z)| - |Af(z)+B(z-w)f'(z)| \tag{2.3}$$

$$\left|\sum_{n=1}^{\infty}(-1)^{n-1}(n+1)a_n(z-w)^n\right| - \left|(B-A)\frac{1}{(z-w)}-\sum_{n=1}^{\infty}(-1)^{n-1}(A+Bn)a_n(z-w)^n\right|$$

or

$$|z-w|\;H(f,f') \le \sum_{n=1}^{\infty}(n+1)\,a_n\,|z-w|^{n+1}-(B-A)+\sum_{n=1}^{\infty}(A+Bn)a_n|z-w|^{n+1}$$

$$=\sum_{n=1}^{\infty}\{(n+1)+(A+Bn)\}\,a_n|z-w|^{n+1}-(B-A)$$

Since the above inequality holds for all $0<|z-w|=r<1$, letting $r\to 1$

we have $\sum_{n=1}^{\infty}\{(n+1)+(A+Bn)\}a_n \le (B-A)$ by (2.1). Hence it follows that $f(z)$ is in the class $\sum_{w,a}^{*}(A,B)$

Corollary 1: If the function $f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n (z-w)^n$, $a_n \geq 0$ is in the class $\sum_{w,a}^{*}(A,B)$ then we have

$$a_n \leq \frac{B-A}{(n+1)+(A+Bn)} \quad , \quad (n \geq 1) \tag{2.4}$$

The result is sharp for the function

$$f_n(z) = \frac{1}{z-w} + (-1)^{n-1} \frac{B-A}{(n+1)+(A+Bn)} (z-w)^n \tag{2.5}$$

3.Distortion properties and Radius of convexity

Theorem 2. If the function $f(z) = \frac{1}{z-w} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n (z-w)^n$, $a_n \geq 0$ is in the class $\sum_{w,a}^{*}(A,B)$,then we have

$$\frac{1}{r} - \frac{B-A}{2+A+B} r \leq |f(z)| \leq \frac{1}{r} + \frac{B-A}{2+A+B} r \tag{3.1}$$

The result is sharp.

Proof: Suppose that $f(z) \in \sum_{w,a}^{*}(A,B)$ By Theorem 1

We have

$$\sum_{n=1}^{\infty} a_n \leq \frac{B-A}{2+A+B} \tag{3.2}$$

Then $|f(z)| = \left| \frac{1}{z-w} + \sum_{n=1}^{\infty} (-1)^{n-1} a_n (z-w)^n \right|$

$$\leq \frac{1}{|z-w|} + \sum_{n=1}^{\infty} a_n |z-w|$$

$$\leq \frac{1}{r} + \frac{B-A}{2+A+B} r$$

Also, $|f(z)| \geq \frac{1}{|z-w|} - \sum_{n=1}^{\infty} a_n |z-w| \geq \frac{1}{r} - \frac{(B-A)}{2+A+B} r$

The result is sharp for the function

$$f(z)=\frac{1}{r}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(B-A)}{2+A+B}r$$

Theorem 3. If the function $f(z)=\dfrac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n(z-w)^n$, $a_n\geq 0$ is in the class $\sum_{w,a}^{*}(A,B)$ then $f(z)$ is meromorphically convex of order δ $(0\leq\delta<1)$ in $|z-w|<r=r(A,B,\delta)$ where

$$r(A,B,\delta)=\inf_{n\geq 1}\left\{\frac{(1-\delta)(n+1)+(A+Bn)}{(B-A)\,n(n+2-\delta)}\right\}^{1/n+1}$$

The result is sharp.

Proof: Let $f(z)$ is in. $\sum_{w,a}^{*}(A,B)$ Then by Theorem 1 we have

$$\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_n\leq 1 \tag{3.3}$$

It is sufficient to show that

$$\left|2+\frac{(z-w)f''(z)}{f'(z)}\right|\leq 1-\delta. \tag{3.4}$$

for $|z-w|\leq r(A,B,\delta)$ where $r(A,B,\delta)$ is specified in the statement of the Theorem. Then

$$\left|2+\frac{(z-w)f''(z)}{f'(z)}\right|=\left|\frac{\sum_{n=1}^{\infty}(-1)^{n-1}n(n+1)a_n(z-w)^{n-1}}{\frac{-1}{(z-w)^2}+\sum_{n=1}^{\infty}(-1)^{n-1}na_n(z-w)^{n-1}}\right|$$

$$\leq\frac{\sum_{n=1}^{\infty}n\,(n+1)a_n|z-w|^{n+1}}{1-\sum_{n=1}^{\infty}n\,a_n|z-w|^{n+1}}$$

This will be bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty}\frac{n(n+2-\delta)}{1-\delta}a_n\,|\,z-w\,|^{n+1}\leq 1 \tag{3.5}$$

By (3.3) it follows that (3.5) is true if

$$\frac{n(n+2-\delta)}{1-\delta}|z-w|^{n+1} \le \frac{(n+1)+(A+Bn)}{(B-A)}, \quad (n \ge 1).$$

or

$$|z-w| \le \left\{\frac{(1-\delta)\{(n+1)+(A+Bn)\}}{(B-A)n(n+2-\delta)}\right\}^{1/(n-1)}, (n \ge 1). \tag{3.6}$$

Setting $|z-w| \le r(A,B,\delta)$ in (3.6),the result follows. The result is sharp for the functions

$$f(z) = \frac{1}{(z-w)} + (-1)^{n-1}\frac{(B-A)}{(n+1)+(A+Bn)}(z-w)^n \tag{3.7}$$

4. CONVEX LINEAR COMBINATIONS

In this section we shall prove that the class $\sum_{w,a}^{*}(A,B)$ is closed under linear combinations.

Theorem4. Let $f_0(z) = \frac{1}{z-w}$ and $f_n(z) = \frac{1}{z-w} + (-1)^{n-1}\frac{(B-A)}{(n+1)+(A+Bn)}(z-w)^n, (n \ge 1).$

Then *f(z)* is in the class $\sum_{w,a}^{*}(A,B)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) \text{ where } \lambda_n \ge 0 \text{ and } \sum_{n=0}^{\infty}\lambda_n = 1$$

Proof: Let $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ with $\lambda_n \ge 0$ and $\sum_{n=0}^{\infty}\lambda_n = 1.$

Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z) = \lambda_0 f_0(z) + \sum_{n=1}^{\infty}\lambda_n f_n(z)$$

$$= \left(\left(1-\sum_{n=1}^{\infty}\lambda_n\right)f_0(z) + \sum_{n=1}^{\infty}\lambda_n f_n(z)\right)$$

$$=\left(1-\sum_{n-1}^{\infty}\lambda_n\right)\frac{1}{z-w}+\sum_{n-1}^{\infty}\lambda_n\left(\frac{1}{z-w}+(1)^{n-1}\frac{B-A}{(n+1)+(A+Bn)}\right)$$

$$=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(B-A)}{(n+1)+(A+Bn)}(z-w)^n$$

Since $\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}\lambda_n\frac{(B-A)}{(n+1)+(A+Bn)}$

By Theorem 1, $f(z)$ is in the class $\sum_{w,a}^{*}(A,B)$

Conversely, suppose that the function *f(z)* is in the class, Since

$$a_n\leq\frac{(B-A)}{(n+1)+(A-Bn)}\ , \qquad n=1,2,3,\ldots\ldots$$

Setting $\lambda_n=\frac{(n+1)+(A+Bn)}{B-A}a_{n},\quad (n\geq 1)$

and $\lambda_0=1-\sum_{n=1}^{\infty}\lambda_n$

It follows that $f(z)=\sum_{n=0}^{\infty}\lambda_n f_n(z).$

This completes the proof of the theorem.

5.Integral Transforms

In this section we consider integral transforms of the functions in $\sum_{w,a}^{*}(A,B)$

Theorem 5. If the function $f(z)=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n(z-w)^n\ ,a_n\geq 0$ is in the class then $\sum_{w,a}^{*}(A,B)$ the integral transforms $F_c(z)=c\int_0^1 uf(uz)du.(0<c<\infty)$ are in the class $\sum_{w,a}^{*}(A,B)$

Proof: Suppose that $f(z)=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n(z-w)^n\ ,a_n\geq 0$

is in $\sum_{w,a}^{*}(A,B)$ Then we have

$$F_c(z)=c\int_0^1 uf(uz)du$$

$$=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^{n-1}\frac{ca_n}{n+c+1}(z-w)^n$$

Since

$$\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}\frac{ca_n}{n+c+1}\le\frac{(n+1)+(A+Bn)}{B-A}a_n\le 1.$$

By Theorem 1, It follows that $F_c(z)$ is in the class $\sum_{w,a}^{*}(A,B)$

Remark 1. In the above theorems**,** putting $w=0$, $A=\beta(2\alpha-1)$ and $B=\beta$

where $0\le\alpha<1$ and $0<\beta\le 1$, we get the results by T. RamReddy,P.ThirupathiReddy and R.B.Sharma [8].

CONVOLUTION PROPERTIES

Robertson [9] has shown that if

$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n z^n$ and $g(z)=\frac{1}{z}+\sum_{n=1}^{\infty}b_n z^n$ are in $\sum_s$ then so is their convolution

$(f*g)(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_n b_n z^n$. We prove the following results for functions in $\sum_{w,a}^{*}(A,B)$

Theorem6. If the functions $f(z)=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^n a_n(z-w)^n$ and $g(z)=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^n b_n(z-w)^n$ are in $\sum_{w,a}^{*}(A,B)$ then $(f*g)(z)=\frac{1}{z-w}+\sum_{n=1}^{\infty}(-1)^{n-1}a_n b_n(z-w)^n$ is in the class $\sum_{w,a}^{*}(A,B)$

Proof: Suppose that $f(z)$ and $g(z)$ are in $\sum_{w,a}^{*}(A,B)$

By Theorem 1, we have

$$\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_n\le 1$$

and $$\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}b_n\le 1$$

Since *f(z)* and *g(z)* are regular in *E,* so is *(f *g)(z).*

Furthermore,

$$\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_nb_n \le \sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_nb_n$$

$$\le \left[\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}a_n\right]\left[\sum_{n=1}^{\infty}\frac{(n+1)+(A+Bn)}{B-A}b_n\right]$$

$$\le 1.$$

Hence by Theorem 1, *(f *g)(z)* is in the class $\sum_{w,a}^{*}(A,B)$.

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