Volume 2,No. 9, September 2014

Journal of Global Research in Mathematical Archives

RESEARCH PAPER

Available online at http://www.jgrma.info

ISSN 2320 – 5822

THE CRANK NICOLSON METHOD COUPLED WITH PROJECTED SUCCESSIVE OVER RELAXATION IN VALUING STANDARD OPTION WITH DIVIDEND PAYING STOCK

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Abstract: We review options pricing with dividend paying stock on a single asset. We start from the Black-Scholes equation with a free boundary value, the free boundary value problem is then transformed into a Linear Complementarity Problem, and an Obstacle Problem. We solve the Linear Complementarity Problem by introducing the method of Finite Difference method - Crank-Nicolson scheme. This leads to a constraint linear system of equations which is solved on a discrete domain by applying the Projected Successive Over Relaxation (PSOR) method. The simulation results showed that the price of the American option exceeds the analytical solution. The payoff function intersects the European option at lower prices relative to the American option; this gives us the early exercise value. We conclude that the American option with dividend paying stock is preferred to the European option.

Keywords: Option Pricing, Crank Nicholson Scheme, Projected Successive Over Relaxation, Black-Scholes Model.

Introduction

An option is the right but not the obligation for a transaction of a risky asset(stocks) at a fixed time for a given price in future. In Dontwi et al. [1], they stated that options are used as valuable tools in numerous hedging strategies as they define the price at which underlying assets can be bought or sold in the future. It gives the holder the right to buy or sell an asset in the future at a price that is agreed upon today. The specified time and the prescribed amount in the contract are the expiration date or maturity and strike price (exercise price) respectively.

Option contract involves two parties; the writer(Bank) and the holder (investor) about trading underlying asset at a certain future time [2]. The holder of the option has the right, not an obligation, to exercise the option. He purchases the option by paying a premium, which is the price (V) of the option. The other party, the writer, who fixes the terms of the contract, has the potential obligation to sell the underlying in case the investor chooses to exercise. As a result, the writer of the option must be compensated for the obligation he assumed, if the investor fails to exercise. The holder is said to be in the long position (buy the option) while the other side of the investor takes the short position(sell the option) of the option contract [3]. The party with the long position agrees to buy the underlying asset while the other party who assumes the short position agrees to sell the asset.

Moreover, the option on stock is said to be exercised when the holder chooses to buy or sell the underlying stock, S. As stated by Wilmott et al. [4] there are two basic types of options; the call and put options. The call option allows the holder the right to buy the underlying for an agreed fixed strike price, K, by maturity date, T. The put option also gives the holder the right to sell the underlying at a certain time, T for an agreed fixed exercise price, K. The exercise rights under option are European and American option. They are not a geographical classification but refers to a technicality in the option contract. Both types are traded in each continent.

Option pricing is widely used amongst academics, practitioners and professionals in the financial market. Over the last 30 years, option pricing on risky assets has long been an intriguing problem as valuation of American option is concerned. It is widely acknowledged that a general closed-form analytical solution does not exist for the American option valuation where early exercise is permitted at anytime during the life of the option, i e where early exercise may be optimal.

In contrast, the European option, which can only be exercised at its maturity date has been valued analytically by the celebrated Black-Scholes formula for the standard financial model [5].

In real markets, many companies pay dividends to the stock holder. The celebrated Black-Scholes model cannot deal with dividend payments, therefore there is the need to extend (modify) the model to include the cash dividends. Since

most traded options are of American type and where solution is based on an iterative method, there is considerable interest in searching for new valuation techniques. In view of that, the valuation of American option routinely resorts to numerical techniques, whose improvement is still an active field of research. Therefore efforts have been concentrated on approximate methods.
It is against this background that the study seeks to employ Finite difference methods (FDM) quite extensively with a focus on the Crank Nicolson method, using transformed Black-Scholes equation (PDE) in valuing standard Option with dividend and apply these numerical techniques to the pricing of standard (vanilla) options.
Morton and Mayers [6] were of the view that, Finite Difference Methods create a mathematical relationship which links together every point on the solution domain, like a chain. The first links in the chain are the boundary conditions and from these, we discover what every other point in the domain has to be. Perhaps the most popular FD methods used in computational finance are: Explicit Euler, Implicit Euler, and the Crank-Nicolson method. However, The main disadvantage to using Explicit Euler is that it is unstable for certain choices of domain discretisation. Though Implicit Euler and Crank-Nicolson involve solving linear systems of equations, they are each unconditionally stable with respect to the domain discretisation. But Crank-Nicolson exhibits the greatest accuracy of the three for a given domain discretisation.
Nwozo and Fadugba [7] pointed out that among the three schemes considered, Crank Nicolson Scheme is unconditionally stable, more accurate and converges faster than binomial model and Monte Carlo Method when pricing standard options, while Monte carlo simulation method is good for pricing path dependent options. Benbow [8] used a Crank-Nicolson finite difference method formulated in a Lagrangian frame in Solving the Black-Scholes equation for the valuation of American options.

Derivation of the Black-Scholes Model (PDE)

In developing the celebrated Black-Scholes model the following assumptions were made in the financial market under consideration, [5]. It is assumed in the Black-Scholes model that

- the stock price follows a log normal random walk in continuous time with a variance rate proportional to the square of the stock price. Thus the distribution of possible stock prices at the end of any finite interval is log-normal.
- The market is frictionless, thus there are no transaction costs (fees or taxes).
- There are no arbitrage possibilities exist, meaning that there are no opportunities of instantly making a risk-free profit.
- The underlying asset pays no dividends during the life of the options.
- The risk-free interest rate r and the variance of the return (volatility) σ are known functions of time over the life of the option.
- The underlying asset trading is continuous and the change of its price is continuous.

If S_t is the price of a security, then the dynamic behaviour of the asset price in a time interval dt can be represented by the SDE given by [9]

$$dS_t = \alpha(S_t, t)dt + \sigma(S_t, t)dW_t \qquad for \quad t \in [0, \infty)$$

where dW_t is an innovation term representing unpredictable events that occur during the infinitesimal interval dt, $\alpha(S_t, t)$ is the drift parameter and $\sigma(S_t, t)$ the diffusion parameter which depends on the level of observed asset price S_t on time t [10].
Following [11] the stochastic process $X = \{X_t, t \geq 0\}$ that solves

$$X_t = X_0 + \int_0^t \alpha(X_s, t)ds + \int_0^t b(X_t, t)dW_s$$

is an Itô process. The corresponding stochastic differential equation is given by

$$dX_t = \alpha(X_t, t)dt + b(X_t, t)dW_t$$

where $\alpha(X_t, t)dt$ is the drift form, $b(X_t, t)dW_t$ is the diffusion form and W_s is a standard Wiener process. If $V(S, t)$ be twice differentiable function of t and of the random process S_t, and S_t follows the Itô`s process

$$dX_t = \alpha_t dt + \sigma_t dW_t, \quad t \geq 0$$

with well behaved drift and diffusion parameters α_t and σ_t then,

$$dV_t = \frac{\partial V}{\partial S_t}dS_t + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}\sigma_t^2 dt.$$

Now,the above conditions lead to an *Itô‘s* stochastic differential equation, describing the behaviour of the asset price which follows a geometric Brownian motion (GBM)

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

where μ denotes the expected return of the underlying asset (drift), σ is the volatility and W follows a Wiener process (Brownian Motion).
We now look for a function $V(S, t)$ that gives the option value for any asset price $S \geq 0$ and at any time $0 \leq t \leq T$. In this setting, $V(S_0, 0)$ is the required time-zero option value. We further assume that such a function exists and is smooth in both variables. Therefore, *Itô‘s* Lemma provides us with a derivative chain rule for stochastic functions. Hence, by *Itô‘s* Lemma

$$df = \frac{df}{dS}(\mu S dt + \sigma S dW) + \frac{1}{2}\sigma^2 S^2 \frac{d^2 f}{dS^2}dt \tag{2}$$

considering equation (2) We write as

$$dV = \sigma S \frac{\partial V}{\partial S}dW + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt. \tag{3}$$

This gives the random walk followed by V . We now construct a portfolio consisting of one option and a proportion $-\Delta$ of the underlying asset. The value of th portfolio is

$$\Pi = V - \Delta S \tag{4}$$

Then the change in the value of this portfolio in one time-step becomes

$$d\Pi = dV - \Delta dS \tag{5}$$

Combining equations (1), (3) and (4) we find that Π follows the random walk

$$d\Pi = \sigma S(\frac{\partial V}{\partial S} - \Delta)dW + (\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S)dt \tag{6}$$

We can eliminate the random component by choosing $\Delta = \dfrac{\partial V}{\partial S}$. This results in a portfolio whose increment is wholly deterministic

$$d\Pi = (\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2})dt \tag{7}$$

The return on an amount Π invested in a riskless asset would see a growth of $r\Pi dt$ in a time dt. If the right hand side of equation (7) were greater than this amount, an arbitrageur could make a guaranteed risk less profit by borrowing an amount Π to invest in the portfolio. Conversely, if the right-hand side of equation (7) were less than $r\Pi dt$ then the arbitrageur would make a risk less, no cost, instantaneous profit. Thus we have

$$r\Pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt \tag{8}$$

Substituting equation(4) into equation(8), where $\Delta = \dfrac{\partial V}{\partial S}$ and dividing by dt. Then we arrive at the Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{9}$$

Any derivative security whose price depends only on the current value of S and on time, t, which is paid for up-front, must satisfy the Black-Scholes equation.
In contrast, the Black-Scholes model discussed above is on assumption that no dividends are paid, where $\lambda = 0$. But when dividend payment is incorporated into the Black-Scholes model, the American options which can be exercised at any time t prior to the maturity date T, leads to the Black-Scholes-inequality. In modelling stock with dividends, the two important questions one needs to asked are:

- When and how often are dividend payments made?

- How large are the dividend payments?

The amounts paid as dividends may be modeled as either deterministic or stochastic. But the focus is on deterministic way only on those equities with dividends whose amount and timing is known at the start of the options life.
Using SDE, equation (1), which is the random walk of the asset price is modified to become

$$dS = (\mu - \lambda)Sdt + \sigma SdW \tag{10}$$

Considering the effect of the dividend payments on our hedged portfolio, we receive an amount $\lambda S\Delta dt$ for every asset held, and since we hold $-\Delta$ of the underlying, the portfolio changes by an amount

$$-\lambda S\Delta dt \tag{11}$$

Adding equations (5) and (11) we obtain

$$d\Pi = dV - \Delta dS - \lambda S\Delta dt \tag{12}$$

Following the same previous analysis, we arrive at

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\lambda)S\frac{\partial V}{\partial S} - rV = 0 \tag{13}$$

Solving the equation (13) for a dividend paying stock using European option, let $\tau = T - t$; where T denotes maturity time, t is current time and τ denotes the remaining life time. The value of European Call and Put options are respectively written as

$$E_c(S,\tau) = Se^{-\lambda\tau}N(d_1) - Ke^{-r\tau}N(d_2) \tag{14}$$

and

$$E_p(S,\tau) = Ke^{-r\tau}N(d_2) - Se^{-\lambda\tau}N(d_1) \tag{15}$$

where $\quad d_1 = \dfrac{\ln(S/K) + (r - \lambda + \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \quad$ and $\quad d_2 = d_1 - \sigma\sqrt{\tau}.$
It is useful to transform the Black-Scholes equation corresponding to (13) into the well known heat-conducting equation to simplify the computation of American options. So we obtain;

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial \tau} \tag{16}$$

for $y(x,\tau), where \quad x \in \mathbb{R}, and \quad \tau \geq 0.$
According to Seydel [2], the equation (16) is a Partial Differential Equation of simplest parabolic type. It can also be written as $y_{xx} = y_\tau$, where y_{xx} is the diffusion term. Both equations (13) and (16) are linear in the dependent variables V or y. The transformation is obtained by applying:

$$\begin{gathered} S = Ke^x, \qquad t = T - \frac{2\tau}{\sigma^2}, \qquad q := \frac{2r}{\sigma^2}, \qquad q_\lambda := \frac{2(r-\lambda)}{\sigma^2} \\ v(x,\tau) := \text{K}\ \exp\{-\tfrac{1}{2}(q_\lambda - 1)x - (\tfrac{1}{4}(q_\lambda-1)^2 + q)\tau\}y(x,\tau) \\ V(S,t) = V(Ke^x, T - \frac{2r}{\sigma^2}) =: v(x,\tau) \end{gathered} \tag{17}$$

In view of the time transformation in equation (17), τ corresponds to the time variable t in the original Black-Scholes equation denotes the remaining life time of the option towards the assuming date: $t = T$ transforms to $\tau = 0$ and $t = 0$ is transformed to $\tau = \frac{1}{2}\sigma^2 T$. And the original domain of the half strip $S > 0$,$0 \leq t \leq T$ of equation (13) becomes the strip

$$-\infty < x < +\infty, \qquad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T$$

on which a solution $y(x,\tau)$ to equation (16) will be approximated. We now apply the transformations of equation (17) to derive out of $y(x,\tau)$ the value of the option $V(S,t)$ in the original variables, after the caculation.
Under the transformations of equation (17), the initial conditions will be

$$\begin{aligned} \text{call} : y(x,0) &= \max\{e^{\frac{x}{2}(q_\lambda+1)} - e^{\frac{x}{2}(q_\lambda-1)}, 0\} \\ \text{put} : y(x,0) &= \max\{e^{\frac{x}{2}(q_\lambda-1)} - e^{\frac{x}{2}(q_\lambda+1)}, 0\} \end{aligned}$$

The payment of dividend lowers the stock price from S to $Se^{\tau-\lambda}$ and the risk- free interest rate which is the rate of return from r to $(r-\lambda)$ according to [3].
Since American option may be exercised at any time prior to the maturity date, exercise under equation (13) is not optimal. The equality sign in equation (13) is replaced by an inequality sign to obtain

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-\lambda)S\frac{\partial V}{\partial S}-rV\leq 0 \tag{18}$$

where V = V(S, t), $S>0$, $0\leq t\leq T$ and V does not depend on μ, but on the riskless interest rate r and the annual dividend yield $\lambda\geq 0$ of the asset [2].
In mathematical literature, $(r-\lambda)S\frac{\partial V}{\partial S}$ is called convection term, $\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}$ is a diffusion term and rV is a reaction term. In this sense, equation (18) is a convection-diffusion PDE. In finance, $\frac{\partial V}{\partial S}$ denotes the option delta (Δ), $\frac{\partial^2 V}{\partial S^2}$ is the option gamma (Γ), and $\frac{\partial V}{\partial t}$ is known as option theta (Θ). [3]. The lower boundary condition for equation (18) is given as

$$\begin{aligned} V_c(S,t)&\geq (S-K)^+ \qquad \text{for all} \quad (S,t),\\ V_p(S,t)&\geq (K-S)^+ \qquad \text{for all} \quad (S,t), \end{aligned} \tag{19}$$

Therefore, the inequalities hold and hence, the value of American options can not be less than their payoff function.
Summarizing all those facts for American options, one obtains the following free boundary-value problems (FBVP):

$$\begin{array}{ll} \textbf{American call option} & \\ \text{for } S<S_f(t): & V(S,t)>(S-K)^+ \text{ and}\\ & V_t+\frac{\sigma^2}{2}S^2V_{SS}+(r-\lambda)SV_S-rV=0\\ \text{for } S>S_f(t): & V(S,t)=S-K \text{ and}\\ & V_t+\frac{\sigma^2}{2}S^2V_{SS}+(r-\lambda)SV_S-rV<0\\ \text{boundary conditions:} & V(0,t)=0\\ & V(S_f(t),t)=S_f(t)-K\\ & V_S(S_f(t),t)=1\\ \text{terminal condition:} & V(S,T)=(S-K)^+ \end{array} \tag{20}$$

$$\begin{array}{ll} \textbf{American put option} & \\ \text{for } S<S_f(t): & V(S,t)=K-S \text{ and}\\ & V_t+\frac{\sigma^2}{2}S^2V_{SS}+(r-\lambda)SV_S-rV<0\\ \text{for } S>S_f(t): & V(S,t)>(K-S)^+ \text{ and}\\ & V_t+\frac{\sigma^2}{2}S^2V_{SS}+(r-\lambda)SV_S-rV=0\\ \text{boundary conditions:} & \lim\limits_{S\to\infty} V(S,t)=0\\ & V(S_f(t),t)=K-S_f(t)\\ & V_S(S_f(t),t)=-1\\ \text{terminal condition:} & V(S,T)=(K-S)^+ \end{array} \tag{21}$$

Note that for American call a dividend yield $\lambda\neq 0$ is needed, because otherwise early exercise of the option is of no advantage to its holder, and the value of the American call equals the European-style call.
this obstacle problem can be equivalently reformulated as a linear complementarity problem:

$$(LCP)=\begin{cases}\text{find a function} \quad u(x) \quad \text{such that :} & \\ u''(u-g)=0, & -u''\geq 0, \quad u-g\geq 0,\\ u(x_0)=u(x_1)=0, & u\in C^1[x_0,x_1]\end{cases} \tag{22}$$

The reverse equivalence is clear, since when a suitable $u(x)$ is found, for a given $g''(x)<0$ one gets the original obstacle problem. Note that in the LCP the boundary values a and b are not mentioned explicitly. However, if one knows a solution for it, the boundaries will also be known.
Recall the free boundary-value problems 20, so they can also be seen as obstacle problems, that is with $u:=V(S,t),g:=(K-S)+$ and $b:=S_f$ for the put. Therefore it is obvious that we can also formulate the evaluation of American options as LCP, where the free boundary S_f is not mentioned explicitly, but will be known when the problem can be solved.
As originally one has to deal with a Black-Scholes inequality when evaluating American options, the direct transformation yields:

$$\frac{\partial^2 y}{\partial x^2}\leq\frac{\partial y}{\partial \tau} \tag{23}$$

for $y(x,\tau)$ with $0\leq\tau\leq\tau_{max}$, $x\in\Re$.
For this inequality, one can construct a linear complementarity problem (LCP) similar to (22). Specifically, the constraints of the FBVP for an American put in equation (20) can also be transformed as LCP. Applying the transformation to them

lead to

$$V_P(S,t) \geq (K-S)^+ = \text{K} \max\{1-e^x, 0\} \tag{24}$$

Inserting this into (17) yields

$$\begin{aligned} y(x,\tau) &\geq \max\{1-e^x,0\}\exp\left\{\frac{1}{2}(q_\lambda-1)x - (\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\} \\ &= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{(1-e^x)e^{\frac{1}{2}(q_\lambda-1)x},0\right\} \\ &= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{e^{\frac{1}{2}(q_\lambda-1)x}-e^{\frac{1}{2}(q_\lambda+1)x},0\right\} =: g(x,\tau) \end{aligned} \tag{25}$$

The terminal condition of the American put, $V(S,T) = (K-S)^+$ implies equality in the above equation, so the initial condition for y is $y(x,0) = g(x,0)$. For $x \to \pm\infty$, we also have $y(x,\tau) = g(x,\tau)$.
With an adjusted function g, it also works for American call with $0 < \lambda < r$.
So formulating the linear complementarity problem

$$\begin{aligned} \text{call} : g(x,\tau) &:= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{e^{\frac{1}{2}(q_\lambda+1)x}-e^{\frac{1}{2}(q_\lambda-1)x},0\right\} \\ \text{put} : g(x,\tau) &:= \exp\left\{(\frac{1}{4}(q_\lambda-1)^2+q)\tau\right\}\max\left\{e^{\frac{1}{2}(q_\lambda-1)x}-e^{\frac{1}{2}(q_\lambda+1)x},0\right\} \end{aligned}$$

$$(LCP)\begin{cases} \text{find a y}(x,\tau)\text{such that :} & \\ \left(\frac{\partial y}{\partial \tau}-\frac{\partial^2 y}{\partial x^2}\right)(y-g)=0 & \\ \frac{\partial y}{\partial \tau}-\frac{\partial^2 y}{\partial x^2} \geq 0, & y-g\geq 0 \\ y(x,0)=g(x,0), & 0\leq \tau \leq \frac{1}{2}\sigma^2 T, \\ y(x,\tau)=g(x,\tau) & \text{for x} \to \pm\infty \end{cases} \tag{26}$$

In financial terms, the heat-conducting inequality (23) means that the expected return from the riskless delta-hedged portfolio is less than the riskless interest rate [4].

Solution To The Black-Scholes Equation

We give the basic description of the ideas of finite differences as they are applied to the equation (16). Each two times continuously differentiable function f satisfies

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{h}{2}f''(\xi)$$

where ξ is an intermediate number between x and $x+h$. The accurate position of ξ is usually unknown. Such expressions are derived by Taylor expansions.
We discretize $x \in \mathbb{R}$ by introducing a one-dimensional grid of discrete points x_i with

$$\cdots < x_{i-1} < x_i < x_{i+1} < \ldots$$

For example, choose an equidistant grid with mesh size $h := x_{i+1} - x_i$. The x is discretized, but the function values $f_i := f(x_i)$ are not discrete, $f_i \in \mathbb{R}$. For $f \in C^2$ the derivative f'' is bounded, and the term $-\frac{h}{2}f''(\zeta)$ can be conveniently written as $O(h)$. This leads to the forward difference with $f \in C^2$

$$f'(x_i) = \frac{f(i+1)-f(i)}{h} + O(h), \tag{27}$$

and the backward difference yields

$$f'(x_i) = \frac{f(i)-f(i-1)}{h} + O(h), \tag{28}$$

Analogous expressions hold for the partial derivatives of $y(x,\tau)$, which includes a discretization in τ . This suggests to replace the neutral notation h by either Δx or $\Delta\tau$, respectively. The fraction in equation (27) is the difference quotient that approximates the differential quotient f' of the left-hand side(LHS); the $O(h^p)$ term is the error. The one-sided (i.e. non-symmetric) difference quotient of equation (27) is of the order $p = 1$. Error orders of $p = 2$ are obtained by central differences

$$f'(x_i) = \frac{f(i+1) - f(i-1)}{2h} + O(h^2) \qquad (\text{for } f \in c^3)$$

$$f''(x_i) = \frac{f(i+1) - 2f(i) + f(i-1)}{h^2} + O(h^2) \qquad (\text{for } f \in c^4) \tag{29}$$

or by one-sided differences that involve more terms, such as

$$f'(x_i) = \frac{-f(i+2) + 4f(i+1) - 3f(i)}{2h} + O(h^2) \qquad (\text{for } f \in c^3)$$

Rearranging terms and indices provides the approximation formula

$$f_i \approx \frac{4}{3}f_{i-1} - \frac{1}{3}f_{i-2} + \frac{2}{3}hf'x_i \tag{30}$$

which is of second order.
Actually, an American option works on the $S-t$ half strip $[0,\infty) \times [0,T]$. But it became an $x-\tau$ strip $(-\infty,\infty) \times [0,\tau_{max}]$ after applying the transformation, where $\tau_{max} := \frac{1}{2}\sigma^2 T$.
Under this section, the domain needs to be discretized to a finite lattice i.e. $[x_{min}, x_{max}] \times [0, \tau_{max}]$. Let Δx and $\Delta\tau$ be the equidistant mesh sizes of the discretizations of x and τ. The choice of the x-discretization is more complicated. So, the infinite interval $-\infty < \times < \infty$ must be replaced by $[x_{min} \leq x \leq x_{max}]$. We chose x_{min} and x_{max} such that the solution on the interval $[x_{min}, x_{max}]$ is in line with the solution on $-\infty < x < \infty$.
For m and ν_{max} be a suitable integers, we define the mesh density by $\Delta x := \frac{x_{max}-x_{min}}{m}$ and the step in τ as $\Delta\tau := \frac{\tau_{max}}{\nu_{max}}$. Since the equidistant of the grid simplifies the implementation and the estimation of the error terms, the work stands better side of it.
The transformation $S = S_i = Ke^{x_i}$, which makes it appropriate to choose $x_{min} < 0$,and $x_{max} > 0$ fit the original limits of the S-interval correctly. The grid is then based on the knots;
$\tau_v := v.\Delta\tau,\quad$ for v $= 0, 1, \ldots, \nu_{max}$
$x_i := x_{min} + i\Delta$x $\quad$ for i $= 0, 1, \ldots, m$
Furthermore, w_i^v denotes the approximation for y_i^v, where $y_i^v := y(xi, \tau_v)$. This is only defined on the discrete nodes and the nodes are the intersection of the points x_i and τ_v. In contrast to the theoretical solution $y(x,\tau)$, y_i^v is defined on a continuum.
The error $\| w_i^v - y_i^v \|$ depends on the prior choice of parameters m, x_{min}, x_{max} and v_{max}. A priori we do not know whose choice of parameters matches a prespecified error tolerance. For instance, if the order of magnitude of these parameters is given by $x_{min} = -5, x_{max} = 5, v_{max} = 100, m = 100$. This choice of x_{min}, x_{max} has shown to be reasonable for a wide range of r, σ-values and accuracies. The actual error is then controlled via the numbers v_{max} and m of grid lines.
With the reference to equation (16), the RHS and LHS of it can be written as equations (27) and (29) respectively to obtained;

$$\frac{\partial}{\partial\tau}y_i^v = \frac{y_i^{v+1} - y_i^v}{\Delta\tau} + O(\Delta\tau), \qquad \text{the forward difference} \tag{31}$$

and

$$\frac{\partial^2}{\partial x^2}y_i^v = \frac{y_{i+1}^v - 2y_i^v + y_{i-1}^v}{\Delta x^2} + O(\Delta x^2), \qquad \text{the central difference} \tag{32}$$

Then the backward difference of the RHS of equation (16) also yields

$$\frac{\partial}{\partial\tau}y_i^v = \frac{y_i^v - y_i^{v-1}}{\Delta\tau} + O(\Delta\tau), \tag{33}$$

With w being the approximation for y, where Δx and $\Delta\tau$ denoted the introduced mesh sizes, we replace equations (31) and (32) into equation (16) and discarding the 0-error terms leads to

$$\frac{w_i^{v+1} - w_i^v}{\Delta\tau} = \frac{w_{i+1}^v - 2w_i^v + w_{i-1}^v}{\Delta x^2} \tag{34}$$

Solving for w_i^{v+1} under the idea of explicit scheme, where all values w are calculated for the time level v, then the values of the time level $(v+1)$ are given by

$$w_i^{v+1} = w_i^v + \frac{\Delta\tau}{\Delta x^2}(w_{i+1}^v - 2w_i^v + w_{i-1}^v)$$

.
Further, starting at $v = 0$, as all $w_i^0 := y(x_i, 0)$, $i = 0, \ldots, m$ are known, each w_i^1 can explicitly be calculated (hence the

name of the method). Then, successively the next levels of time can be proceeded, for $1 \leq v \leq v_{max}$.
With the notation $\zeta := \frac{\Delta\tau}{\Delta x^2}$, the result is written compactly in time-iteration form as

$$w_i^{v+1} = \zeta w_{i+1}^v + (1-2\zeta)w_i^v + \zeta w_{i-1}^v. \tag{35}$$

The total error is $O(\Delta\tau + \Delta x^2)$ for $y \in C^{4,2}(\bar{D}_w)$, for $D_w := (x_{min}, x_{max}) \times (0, \tau_{max})$.
This method is sometimes called implicit method. But to distinguish it from other implicit methods, we call it fully implicit, or backward-difference method, or more accurately, backward time centered space scheme (BTCS).
Using the backward difference, equation (32) and equation (33) to discretize the heat-conducting equation, (16), yields

$$\frac{w_i^v - w_i^{v-1}}{\Delta\tau} = \frac{w_{i+1}^v - 2w_i^v + w_{i-1}^v}{\Delta x^2}$$

This is rewritten as

$$\frac{w_i^{v+1} - w_i^v}{\Delta\tau} = \frac{w_{i+1}^{v+1} - 2w_i^{v+1} + w_{i-1}^{v+1}}{\Delta x^2}, \tag{36}$$

with the same 0-error terms as in the explicit scheme. Sorted by time-levels, we obtain the iteration form

$$-\zeta w_{i+1}^{v+1} + (2\zeta+1)w_i^{v+1} - \zeta w_{i-1}^{v+1} = w_i^v \tag{37}$$

The equation (37) couples three unknowns. Therefore, only the value w_i^v of the RHS of the equation (37) is known, whereas on the LHS of the same equation in each step, one has to compute three unknown variables.
Eventually this leads to a linear system of equations (LSE) that includes all time stages. This system can then be solved.
The method is unconditionally stable for all $\Delta\tau > 0$
The summation of equations (34) and (36), and truncating the error terms yields

$$\frac{w_i^{v+1} - w_i^v}{\Delta\tau} = \frac{w_{i+1}^v - 2w_i^v + w_{i-1}^v + w_{i+1}^{v+1} - 2w_i^{v+1} + w_{i-1}^{v+1}}{2\Delta x^2} \tag{38}$$

With $\zeta := \frac{\Delta\tau}{\Delta x^2}$ the equation (38) can be rewritten as

$$-\frac{\zeta}{2}w_{i-1}^{v+1} + (1-\zeta)w_i^{v+1} - \frac{\zeta}{2}w_{i+1}^{v+1} = \frac{\zeta}{2}w_{i-1}^v + (1-\zeta)w_i^v + \frac{\zeta}{2}w_{i+1}^v. \tag{39}$$

To get the error for the CN scheme for a $y \in C^{4,3}(\bar{D}_w)$, D_w defined as before, first consider the L.H.S. of 38 by using the first three terms of the Taylor expansion, it can be approximated by

$$\frac{(w(v-\Delta\tau) - w(v))}{\Delta\tau} = w_\tau + \frac{1}{2}w_{\tau\tau}\Delta\tau + O((\Delta\tau)^2)$$

From the R.H.S. of equation (38) it follows

$$\frac{1}{2}(w_{xx}(x,\tau) + w_{xx}(x,\tau+\Delta\tau)) = \frac{1}{2}(2w_{xx} + w_{xx\tau}\Delta\tau + O((\Delta x)^2 + (\Delta\tau)^2))$$

Eventually, we get the total consistency error

$$c_{err} = w_\tau - w_{xx} + \frac{1}{2}\Delta\tau(w_{\tau\tau}) - w_{xx\tau} + O((\Delta x)^2 + (\Delta\tau)^2)$$

The Crank-Nicholson approach has got a better order than the former two methods. But similar to the one before, there is no explicit way to solve (39). Again, one needs to set up an LSE, which then can be evaluated. Thus, the CN scheme is also of implicit type.
The iteration of the three FD schemes i.e. equations (35), (37) and (39) can be written as a sequence of LSEs, that is

$$Aw^{v+1} = Bw^v + c^v, \qquad v = 0, \ldots, v_{max} - 1 \tag{40}$$

with the matrices

$$A = \begin{pmatrix} 1+2\zeta\theta & -\zeta\theta & & 0 \\ -\zeta\theta & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}, \; B = \begin{pmatrix} 1-2\zeta\tilde{\theta} & \zeta\tilde{\theta} & & 0 \\ \zeta\tilde{\theta} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

The two matrices are tridiagonal and of the dimension $(m-1)\times(m-1)$. The free paremeters θ and $\tilde{\theta} := 1-\theta$ denote the particular FD scheme, where for; $\theta = 0$, explicit, $\theta = \frac{1}{2}$, Crank-Nicolson $\theta = 1$, implicit.
$\zeta = \frac{\Delta\tau}{\Delta x^2}$, as defined before. The vectors are $w^i := (w_1^i, \ldots, w_{m-1}^i)^T, i = \{v, v+1\}$ and $c^v := (c_0^v, 0, \ldots, 0, c_m^v)^T$. The elements c_0^v and c_m^v contain the terms that were discarded when setting up matrix A and B. In particular, they are defined by the boundary conditions of the PDE. Note that the actual setup of the matrices A and B depends both on m and v_{max}, where the former parameter influences the size, and both of them affect the eigenvalues.
A uniform solution of the system of equations (40) exists, if matrix A has an inverse, which is actually true for $\zeta > 0$ and $\theta \in [0,1]$. To prove this statement, we need to show that no eigenvalue λ of A equals zero:
Let $x := (x_1, \ldots, x_n)^T$ be an arbitrary eigenvector of A with the corresponding eigenvalue λ. Let $x_i := max\{|x_j| : x_j \text{ is element of} x\}, i, j = \{1, \ldots, n\}$. Then, from $\lambda x_i = (Ax)_i = \sum\limits_{j=1,\ldots,n} a_{ij}x_j$, after division by $x_i \neq 0$ follows:

$$|\lambda - a_{ii}| = \left|\sum_{j\neq i} a_{ij}\frac{x_j}{x_i}\right| \leq \sum_{j\neq i}|a_{ij}|$$

So by claim of Gerschgorin's Theorem, for the eigenvalues of A as in (40), one has

$$1 \leq \lambda \leq 1 + 4\zeta\theta,$$

with ζ and θ defined as before, and in particular,$\lambda \leq 0$, q.e.d.
Following [2] Finite Differences are an efficient tool to solve the parabolic equation. Recall from (26), that in order to evaluate American options, we actually need to solve a LCP, containing a heat-inequality. This means, that the iteration (40) needs to be adjusted to

$$Aw^{(v+1)} \geq Bw^{(v)} + c^{(v)},\ v = 0, \ldots, v_{max} - 1 \tag{41}$$

Additionally, the LCP claims $y - g \geq 0$, which in terms of the FD discretization leads to $w^{(v)} \geq g^{(v)}$. Note that inequalities in vectors are meant to be component-wise.
The last things missing are the initial conditions (w^0), and the structure of vector c, which is defined by the boundary conditions. From the LCP (26) we get $w_i^0 = g_i^0$ for $i = 1, \ldots, m-1$, i.e. $w^0 = g^0$. The boundary conditions are $w_0^{(v)} = g_0^{(v)}$ and $w_m^{(v)} = g_m^{(v)}$ for all $v \geq 1$, yielding

$$c^{(v)} := \begin{pmatrix} \zeta\theta g_0^{(v+1)} + \zeta\tilde{\theta} g_0^{(v)} \\ 0 \\ \vdots \\ 0 \\ \zeta\theta g_m^{(v+1)} + \zeta\tilde{\theta} g_m^{(v)} \end{pmatrix} \tag{42}$$

Since in step (v+1) the R.H.S. of (41) is completely known, we can set

$$b := Bw^{(v)} + c^{(v)}, \tag{43}$$

and rewrite the LCP (26):

$$\begin{cases} \text{find } w := w^{(v+1)}, \text{ such that} \\ Aw \geq b,\ w \geq g,\ (w-g)^T(Aw - b) = 0 \end{cases} \tag{44}$$

There are many ways to solve a linear system of equations with numerical methods. However, in our context, when dealing with large sparse matrices (A), iteration methods are of advantage, compared to ordinary elimination schemes, since they require less memory and arithmetic cost.
The main idea to solve a linear system of equations by fixed point iteration is to choose a suitable regular matrix $Q \in \mathbb{R}^{n\times n}$, then

$$Qx = (Q - A)x + b \tag{45}$$

$$x = (I - Q^{-1}A)x + Q^{-1}b = \Phi(x) \tag{46}$$

Now let $M = (I - Q^{-1}A)$ and $c = Q^{-1}b$, then

$$x^{k+1} = \Phi(x^{(k)}) = Mx^{(k)} + c. \tag{47}$$

Equation (47) converges if and only if $\rho(M) < 1$, where $\rho(M)$ is the spectral radius of M. In order to optimize convergence, we require the matrix M in an appropriate manner. Therefore, we decompose A into $A = D - L - U$, where D contains the diagonal, L and U are the lower and the upper elements of A, respectively. We now focus on Relaxation Methods.

Let us choose

$$Q := \frac{1}{\omega}D - L \quad \Longrightarrow M_\omega = I - (\frac{1}{\omega}D - L)^{-1}A,$$

with a relaxation parameter $\omega > 0$. This leads to the iteration

$$x^{(k+1)} = \left(I - \left(\frac{1}{\omega}D - L\right)^{-1} A\right) x^{(k)} + \left(\frac{1}{\omega}D - L\right)^{-1} b. \tag{48}$$

When $\omega = 1$ then equation (48) equals the Gauss-Seidel method. For $0 < \omega < 1$, the iteration is called damped, and for $1 < \omega < 2$ the scheme is called Successive Over-relaxation (SOR). In practice, we can approach the iteration by rewriting (48) equivalently as

$$\begin{aligned}
\frac{1}{\omega}(D - L)\, x^{(k+1)} &= \left(\left(\frac{1}{\omega}D - L\right) - A\right) x^{(k)} + b \qquad \Longleftrightarrow \\
\frac{1}{\omega}Dx^{(k+1)} &= Lx^{(k+1)} + \left(\left(\frac{1}{\omega}D - L\right) - D + L + U\right) x^{(k)} + b \qquad \Longleftrightarrow \\
x^{(k+1)} &= x^{(k)} + \omega D^{-1}(Lx^{(k+1)} - Dx^{(k)} + Ux^{(k)} + b),
\end{aligned}$$

and assuming that for step $x^{(k+1)}$ the components $x_i^{(k+1)}$, $1 \leq i \leq j-1$, are already known.
We will focus on an extension of the SOR method given as:

$$\begin{aligned}
w - A^{-1}b \geq b, \;\; w - g \geq g, \;\; (w-g)^T(Aw - b) = 0 \quad &\Longleftrightarrow \\
\underset{w}{min}\{w - A^{-1}b,\; w - g\} = 0 \quad &\Longleftrightarrow \\
w = max\{A^{-1}b,\; g\}
\end{aligned}$$

Analysis and Results

An American put option is evaluated with the following financial values. The strike price (K) is chosen as 50. An annual volatility (σ) equal 0.6. The risk free interest rate (r) and the maturity time of underlying(T) are both 0.25 and 1 respectively. The tolerance parameter (ε) is chosen as 10^{-6}. This is due to the fact that PSOR iteration, the accuracy of the convergence test depends on the tolerance parameter. We choose the relaxation parameter as 1.15. These benchmark parameter are randomly chosen for the purpose of the analysis to befit existing literature values.

A Call Option with Dividend Paying Stock

Now examining the differences between the European call, the payoff function and the American call option with dividend paying stock, we considered dividend λ value of 0.2 is used for the computation. The price of the underlying (S) and the strike price (K) are both chosen as 80. An annual volatility (σ) is chosen as 0.6. The risk free interest rate (r) and the maturity time of underlying(T) are chosen as 0.25 and $\frac{5}{12}$ respectively.
Though the pictorial view depict same prices for the American call, the European call and the payoff function when the strike price is less than thirty. There are slight differences between the corresponding numerical values. The payoff function is zero when the price of the underlying asset is less than the strike price. The payoff function rises immediately the price of the underlying asset becomes greater than the strike price.

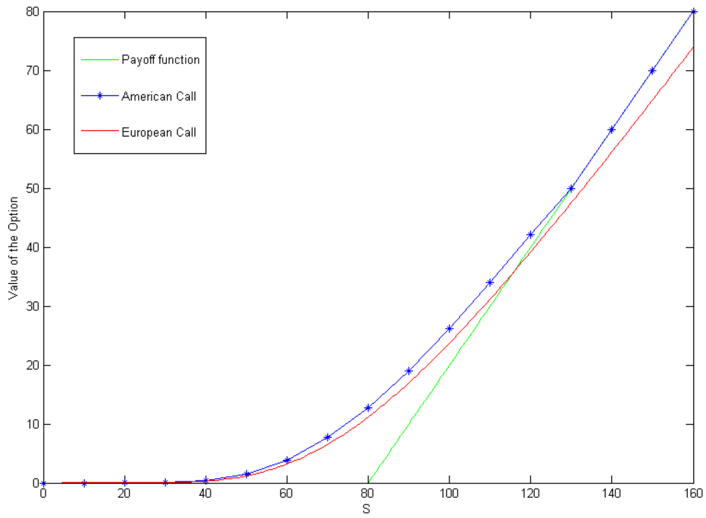

Figure 1: The value function $V(S,0)$ of the European and American call when $K = 80, \quad \sigma = 0.6, \ \lambda = 0.2, \ r = 0.25, \ T = \frac{5}{12}, \ S = 80.$

The payoff function intersect both the European call and the American call functions. But, it intersects the European call at lower call price with it relativity to the American option. The PSOR simulation of the American call with dividend paying stock more desirable than the European call. A similar analysis of a put option with dividend paying stock is shown in Figure 2.

A put option with dividend paying stock

An American put option with dividend paying stock is evaluated with the following financial values. The strike price $(K) = 80$, an annual volatility (σ) equal 0.6. The risk free interest rate $(r) = 0.25$, the maturity time of underlying, measured in years $(T) = 1$. $S = 80$, that is, the asset value price at issuing date and dividend $\lambda = 0.2$.
Below shows a clear cut distinction between American put dividend paying stock, the pay-off function and the European put. The blue line represents for the American put option, the green line for the European put, and the pay-off function by the red line. The pay-off function has an intersection with both the American and European put. Figure 2 presents a put option on which dividend are paid.

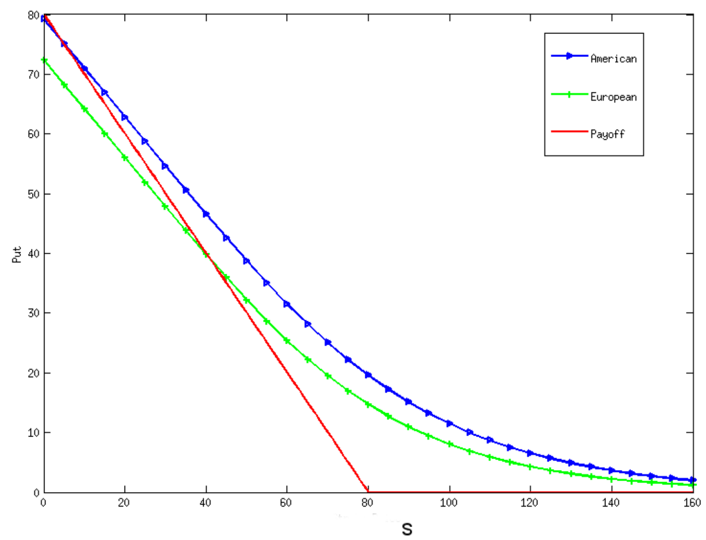

Figure 2: The value function $V(S,0)$ of the European and American put when $K = 80, \ \sigma = 0.6, \ r = 0.25, \ T = 1, \ S = 80.$

Convergence to the Exact Solution

The following financial values were employed for the Numerical Simulation in Table 1. The strike price $(K) = 10$, an annual volatility (σ) equal 0.6, dividend $\lambda = \frac{2}{10}$. The risk free interest rate $(r) = 0.1$, the maturity time of underlying, measured in years $(T) = \frac{5}{12}$. $S = 10$, that is, the asset value price at issuing date.

Table 1: Value of the option as the mesh size M is varied

100	200	300	400	500	600	700	800	900	1000	2000	3000	4000	5000
26.9890	27.0059	27.0090	27.0101	27.0106	27.0109	27.0111	27.0112	27.0112	27.0112	27.0112	27.0112	27.0112	27.0112

As the mesh size for the PSOR iteration $V_{max} = M$ increases, the price of the option converges to the exact solution. There are no changes in the price of the option as the value of mesh size goes beyond 800. All higher values of mesh size has an option value of 27.0115. Hence, the exact solution of the option is 27.0115.

Option at Early Exercise

An American call option with dividend paying stock, where early exercise exist is evaluated with the strike price $(K) = 80$, an annual volatility (σ) equal 0.6. The risk free interest rate $(r) = 0.25$, the maturity time of underlying, measured in years $(T) = 1$. The current price was issued at $S = 80$ attracting dividend (λ) of 0.2.

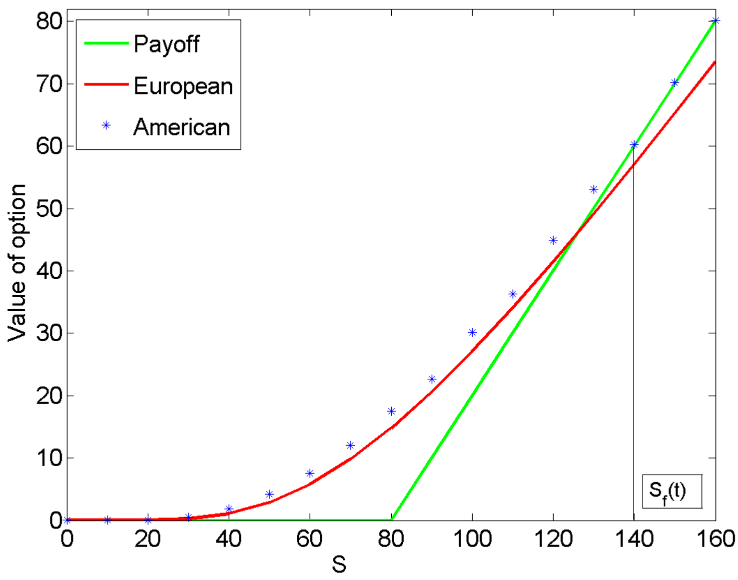

Figure 3: The value function $V(S, 0)$ of the European and American call when $K = 80,\ \sigma = 0.6,\ r = 0.25,\ T = 1,\ \lambda = 0.2,\ S = 80$.

At a price $S_f(t)$, the American call option behaves almost identically to the pay-off function which implies that early exercise is possible. This means that the option value tangentially touches the pay-off function in $S_f(t)$. As long as the the option value, $V(S, t)$ coincides with the pay-off function, a financial investor executes the option as early as possible to maximize profit.

Furthermore, at the point when S is less than $S_f(t)$ i.e. ($S < S_f(t)$), the holder will retain or hold the option and allow it goes worthless. Conversely, the holder will exercise the call option and make profit if the spot price S is greater than or equal to $S_f(t)$. This means that if a call is executed, K purchased stock is sold for S and the profit $S - K$ should be invested in a risk-less asset.

Therefore, early exercise becomes possible at a stock price of 139.0154. This is shown by the position of the free boundary point $S_f(t)$ in Figure 3 above.

Effect of Varied Interest rate

With the dividend rate, λ given as 0.2. The price of the underlying (S) and the strike price (K) are both chosen as 80. An annual volatility (σ) is chosen as 0.6. The maturity time of underlying (T) is 1. The asset matures at the end of the years.

The price of the option at 10% interest rate for all values of the strike price are much higher than at 30%, 60% and 80% interest rate. As the rate of interest rises, the price of American put option falls. Figure 4 demonstrated this distinct values of interest rate.

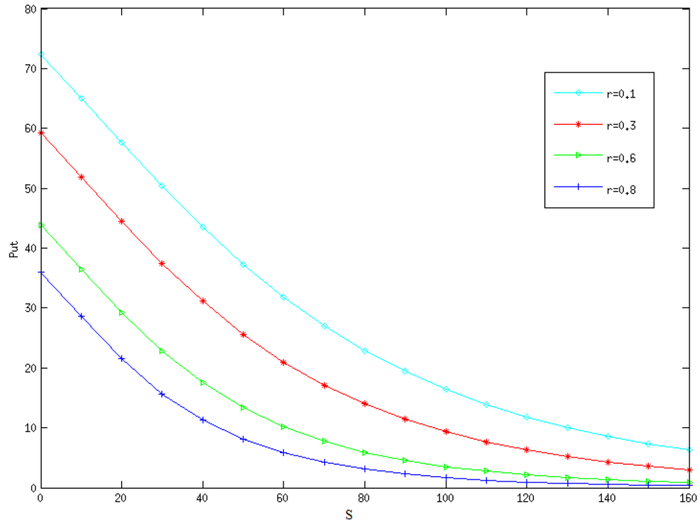

Figure 4: Value of American put option with interest rate values of 10%, 30%, 60%, and 80%

Value of the Option with varying annual volatility

American put option are evaluated with the following financial values. The risk free interest rate (r) and the maturity time of underlying(T) are both 0.25 and $\frac{5}{12}$ respectively. The asset price at issuing date, $S = 80$. The strike price (K) is chosen as 80 and dividend $\lambda = 0.2$. We compute this based on different annual volatility values.

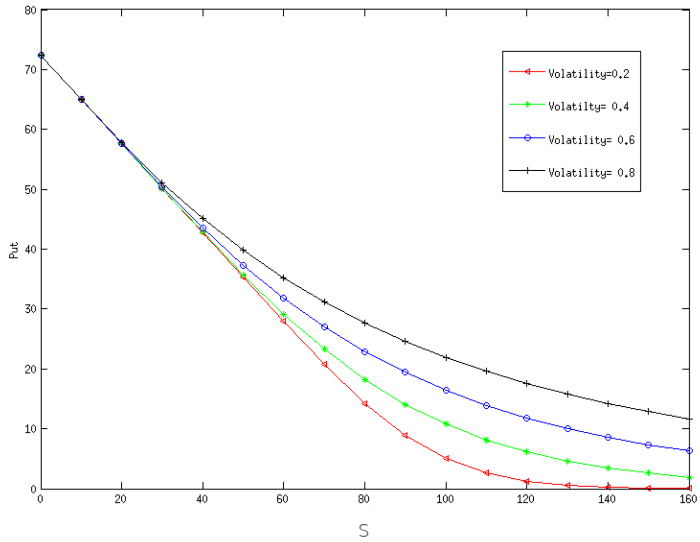

Figure 5: An American put option with selected annual values for volatility

The volatility rates for this simulation are 0.2, 0.4, 0.6, and 0.8. This is indicated in Figure 5. Within a particular range of the strike price, the prices of the American put remains same for all values of volatility selected for the simulation. At the latter part, the disparity between the prices of the option is shown clearly.

Different Maturity Times Measured in Years

Quarterly time intervals in years are chosen for computing the value of the on American call option. The following financial values were employed for the calculation in Figure 6. The strike price $(K) = 50$, an annual volatility (σ) equal 0.6, dividend $\lambda = \frac{2}{10}$. The risk free interest rate $(r) = 0.25$, $S = 10$, that is, the asset value price at issuing date.

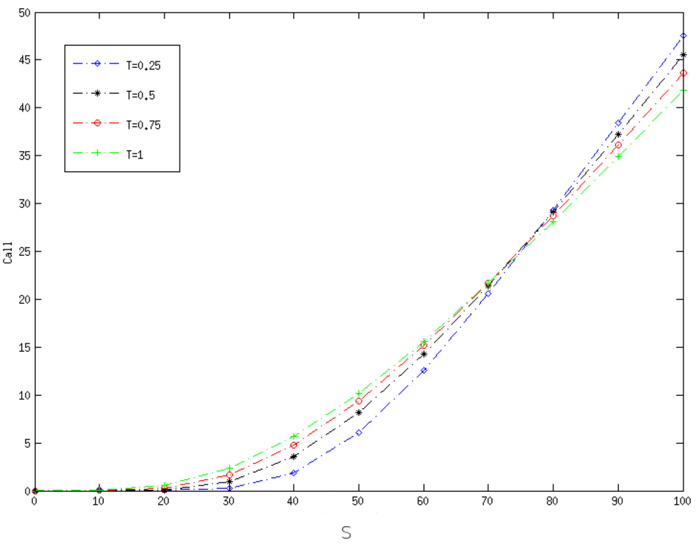

Figure 6: American call option when $\lambda = 0.2,\ K = 10,\ \sigma = 0.6,\ r = 0.1,\ T = \frac{5}{12},\ \lambda = 0.2,\ S = 10.$

For American call with dividend paying stock, the early exercise value remains the same irrespective of the time of maturity. This is shown by the point of intersections in Figure 6. Maturity time is an important element to consider when dealing with American option.

Conclusion

In this study, we applied the Crank Nicolson method in valuing standard option with dividend paying stock. Different financial values were chosen for the computation. We evaluated for a call option at different dividends values. It was realized that the price of the option has negative relationship with the dividend value.
For the dividends paying stocks, the computational results for the prices of the American option exceed the analytical solution. The pay-off function intersects both the European call and the American call functions. But, it intersects the European call at lower call price compared to the American option. The PSOR simulation of the American call with dividend paying stock is more desirable than the European call. For the put option, the American option is also favourite. For early exercise, the holder has to exercise the call option and make profit when the spot price S is greater than or equal to $S_f(t)$. This means that, if a call is executed, K purchased stock is sold for S and the profit $S - K$ should be invested in a risk-less asset.

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