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TERMINAL ZAGREB ECCENTRICITY INDICES OF LINE GRAPHS

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*Abstract***:** Terminal Zagreb eccentricity indices were proposed analogously to Zagreb eccentricity indices. For a connected graph, the first Terminal Zagreb eccentricity index is defined as the sum of the squares of the eccentricities of the terminal vertices, and the second Zagreb eccentricity index is defined as the sum of the products of the eccentricities of all pairs of terminal vertices. In this paper we obtain results for the terminal Zagreb eccentricity indices of line graphs.
KEYWORDS: Zagreb eccentricity index, Terminal Zagreb eccentricity index, Line graph.

1. INTRODUCTION

Let G be a connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. The *degree* of a vertex v in G is the number of edges incident to it and is denoted by $d(v)$ or $deg_G(v)$. If degree of v is one then v is called a *pendent vertex* or *terminal vertex*. An edge $e = uv$ of a graph G is called a *pendent edge* if $d(u) = 1$ or $d(v) = 1$. The *line graph* of a connected graph G, denoted by $L(G)$ is the graph whose vertices are the edges of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent. The *distance* between the vertices v_i and v_j in G is equal to the length of a shortest path joining them and is denoted by $d(v_i, v_j /G)$. For a vertex v_i its *eccentricity*, e_i is the largest distance from v_i to any other vertices of G.

The first and the second Zagreb eccentricity indices [6, 9] are defined as follows,

$$E_1 = E_1(G) = \sum_{v_i \in V(G)} e_i^{\,2} \qquad \text{….. (1.1)}$$

$$E_2 = E_2(G) = \sum_{v_i v_j \in V(G)} e_i e_j \qquad \text{….. (1.2)}$$

Analogously to Zagreb eccentricity indices, defining the first and the second Terminal Zagreb eccentricity indices as,

$$T[E_1(G)] = \sum_{v_i \in V_T(G)} e_i^{\,2} \qquad \text{….. (1.3)}$$

$$T[E_2(G)] = \sum_{v_i v_j \in V_T(G)} e_i . e_j \qquad \text{….. (1.4)}$$

Where, $V_T(G) = \{v_1, v_1, \ldots, v_k\}$ is the set of all pendent vertices of G.

Defining the set $D_2(G)$ as,

$D_2(G) = \{v \mid deg_G(v) = 2$ and one neighbour of v is pendent$\}$.

2. EXISTING RESULTS

Many researchers have studied and obtained several results on Zagreb eccentricity indices of various graphs [1, 2, 3, 4, 5, 7, 10]

Recently H. S. Ramane et. al. [8] have obtained expressions for terminal weiner index of Line graphs

Theorem 2.1 [8]: Let G be a connected graph with $n \geq 4$ vertices and let $D_2(G) = \{v_1, v_2, \ldots, v_q\}$. Then

$$TW(L(G)) = \sum_{1 \leq i < j \leq q} d(v_i, v_j / G) + \frac{q(q-1)}{2}$$

Corollary 2.2 [8]: $TW(L(G)) = 0$ if and only if the graph G satisfies one of the following conditions. (*i*) G has no edge $e = uv$ where $deg_G(u) = 1$ and $deg_G(v) = 2$. (*ii*) G has only one edge $e = uv$ where $deg_G(u) = 1$ and $deg_G(v) = 2$ (*iii*) G has no pendent vertices. (*iv*) G has only one pendent vertex. (*v*) G has no vertex of degree 2.

Theorem 2.3 [8]: Let G be a connected graph with $n \geq 4$ vertices and G' be the graph obtained from G by removing pendent vertices of G. If p is the number of pendent vertices of G', then

$$TW(L(G)) \leq TW(G') + \frac{p(p-1)}{2}.$$

Equality holds if and only if (*i*) $G = K_{1,\, n-1}$ or (*ii*) G has no Bridge e such that one of the component of $G - e$ is $K_{1,\, s}$, $s \geq 2$ and $G \neq K_{1,\, n-1}$.

Corollary 2.4 [8]: Let G be a connected graph with $n \geq 4$ vertices and G' be the graph obtained from G by removing pendent vertices. Let p be the number of pendent vertices of G'. If all pendent edges of G are mutually independent, then

$$TW(L(G)) = TW(G') + \frac{p(p-1)}{2}.$$

3. TERMINAL ZAGREB ECCENTRICITY INDICES OF LINE GRAPHS

Theorem 3.1: Let G be a connected graph with $n \geq 4$ vertices and $D_2(G) = \{v_1, v_2, \ldots, v_q\}$. Then the terminal Zagreb first and second eccentricity index of line graph of G is,

$$T[E_1(L(G))] = \sum_{v_i \in D_2(G)} e_i^{\,2} \text{ and } T[E_2(L(G))] = \sum_{v_i v_j \in D_2(G)} e_i . e_j .$$

Proof: Let G be a connected graph with $n \geq 4$ vertices and $D_2(G) = \{v_1, v_2, \ldots, v_q\}$.
Let $E_k = \{e_1, e_2, \ldots, e_k\}$ be the set of pendent edges of G. We know that if $e_i = uv \in E_q$,
where $E_q \subseteq E_k$ then $deg_G(u) = 1$ and $deg_G(v) = 2$, $i = 1,2,\ldots,q$.
Consider two edges e_i and e_j with $e_i = uv \in E_q$ and $e_j = v_j w \in E_q$.
Where, $deg_G(u) = 1 = deg_G(w)$ and $deg_G(v) = deg_G(v_j)$, $i = 1, 2, \ldots, q$.
Therefore e_i and e_j are the pendent vertices of $L(G)$
Therefore,

$$T[E_1(L(G))] = \sum_{vi \in D_2(G)} e_i^{\,2} \text{ and } T[E_2(L(G))] = \sum_{v_i v_j \in D_2(G)} e_i . e_j . \qquad \square$$

Corollary 3.2: $T[E_1(L(G))] = T[E_2(L(G))] = 0$ if and only if the graph G satisfies one of the following conditions. *i*) G has no edge $e = uv$ where $deg_G(u) = 1$ and $deg_G(v) = 2$. (*ii*) G has only one edge $e = uv$ where $deg_G(u) = 1$ and $deg_G(v) = 2$. (*iii*) G has only one pendent vertex. (*iv*) G has no pendent vertices. (*v*) G has no vertex of degree 2.

Theorem 3.3: Let G be a connected graph with $n \geq 4$ vertices and G' be the graph obtained from G by removing pendent vertices of G. If p is the number of pendent vertices of G', then

$$T[E_1(L(G))] \leq \sum_{vi \in D_2(G)} e_i^{\,2} + \sum_{vj \in V_T(G')} e_j^{\,2} \text{ and } T[E_2(L(G))] \leq \sum_{\{v_i, v_j\} \subseteq D_2(G)} e_i .\, e_j + \sum_{\{v_k, v_l\} \subseteq V_T(G')} e_k .\, e_l$$

Equality holds if and only if (*i*) $G = k_{1,n-1}$ or (*ii*) G has no bridge e such that one of the component of $G–e$ is $k_{1,s}$, $s \geq 2$ and $G \neq k_{1,\, n-1}$.

Proof: Let $D_2(G) = \{v_1, v_2, v_3, \ldots, v_q\}$. The number of pendent vertices of G' is at least q. If p is the number of pendent vertices in G' then $p \geq q$. from Theorem 3.1,

$$T[E_1(L(G))] = \sum_{vi \in D_2(G)} e_i^{\,2} \leq \sum_{vi \in D_2(G)} e_i^{\,2} + \sum_{vj \in V_T(G')} e_j^{\,2} \text{ and}$$

$$T[E_2(L(G))] = \sum_{\{v_i, v_j\} \subseteq D_2(G)} e_i .\, e_j \leq \sum_{\{v_i, v_j\} \subseteq D_2(G)} e_i .\, e_j + \sum_{\{v_k, v_l\} \subseteq V_T(G')} e_k .\, e_l$$

Therefore,

$$T[E_1(L(G))] \leq \sum_{vi \in D_2(G)} e_i^{\,2} + \sum_{vj \in V_T(G')} e_j^{\,2} \text{ and } T[E_2(L(G))] \leq \sum_{\{v_i, v_j\} \subseteq D_2(G)} e_i .\, e_j + \sum_{\{v_k, v_l\} \subseteq V_T(G')} e_k .\, e_l$$

For equality we consider the following cases:
Case I*:* for $G = K_{1,\, n\text{-}1}$ obviously equality holds .
Case II*:* If $G \neq K_{1,\, n\text{-}1}$ and if there is an edge in G such that one of the component of $G–e$ is $K_{1,\, s}$, $s \geq 2$ then $q = p$.
Therefore

$$\sum_{v_i \in D_2(G)} e_i^{\,2} = \sum_{v_j \in V_T(G')} e_j^{\,2} \text{ and } \sum_{\{v_i, v_j\} \subseteq D_2(G)} e_i .\ e_j = \sum_{\{v_k, v_l\} \subseteq V_T(G')} e_k .\ e_l$$

i.e., $T[E_1(L(G))] = T[E_1(L(G'))]$ and $T[E_2(L(G))] = T[E_2(L(G'))]$

Conversely, let G contains a bridge e such that one of the component of $G –e$ is $K_{1,\, s}$, $s \geq 2$
Therefore $p > q$.

$$\sum_{vi \in D_2(G)} e_i^{\,2} < \sum_{vj \in V_T(G')} e_j^{\,2} \qquad \text{….. (3.1)}$$

From Theorem 3.1

$$T[E_1(L(G))] = \sum_{vi \in D_2(G)} e_i^{\,2} < \sum_{vj \in V_T(G')} e_j^{\,2} \text{ and } T[E_2(L(G))] = \sum_{\{v_i, v_j\} \subseteq D_2(G)} e_i .\ e_j < \sum_{\{v_k, v_l\} \subseteq V_T(G')} e_k . e_l$$

By Eq. (3.1), $q < p$

$T[E_1(L(G))] < T[E_1(L(G'))]$ and $T[E_2(L(G))] < T[E_2(L(G'))]$

Which is a contradiction.
This completes the proof. □

Corollary 3.4: Let G be a connected graph with $n \geq 4$ vertices and G' be the graph obtained from G by removing pendent vertices of G. If p is the number of pendent vertices of G'. If all pendent edges of G are mutually independent, then
$T[E_1(L(G))] = T[E_1(G')]$ and $T[E_2(L(G))] = T[E_2(G')]$

Proof: Follows from the equality part of Theorem 3.3. □

4. Terminal Zagreb eccentricity index of line graphs of some graphs

Let the vertices of G be $v_1, v_2, \ldots, v_n$ then G^+ is the graph obtained from G by adding n new vertices $v'_1, v'_2, \ldots, v'_n$ and joining v'_i to v_i an edge, $i = 1, 2, \ldots, n$.

Theorem 4.1: Let G be a connected graph with k pendent vertices, then

$$T[E_1(L(G^+))] = \sum_{vi \in D_2(G^+)} e_i^{\,2} \text{ and } T[E_2(L(G))] = \sum_{\{v_i, v_j\} \subseteq D_2(G^+)} e_i .\ e_j .$$

Proof: If G has n vertices of which k are pendent vertices, then G^+ has n pendent edges of which k pendent edges are such that for each $e_i = uv$, $i = 1, 2, \ldots, k$.

$\deg_{G^+}(u) = 1$ and $\deg_{G^+}(v) = 2$

Therefore $deg_{L(G^+)}(u) = deg_{G^+}(u) + deg_{G^+}(v) - 2 = 1.$

$L(G^+)$ has k pendent vertices.
We know that pendent vertices of G^+ are mutually independent, from Corollary 3.4,

$$T[E_1(L(G^+))] = \sum_{vi \in D_2(G^+)} e_i^{\,2} \text{ and } T[E_2(L(G))] = \sum_{\{v_i, v_j\} \subseteq D_2(G^+)} e_i . e_j .$$ □

Theorem 4.2: $T[E_1(L(S_n^+))] = T[E_1(K_{n-1}^+)]$ and $T[E_2(L(S_n^+))] = T[E_2(K_{n-1}^+)]$.

Proof: Star graph S_n has $n – 1$ pendent vertices S_n^+ has n pendent vertices and $L(S_n^+)$ has $n – 1$ pendent vertices. Therefore terminal Zagreb eccentric indices of $L(S_n^+)$ is same as the terminal Zagreb eccentric indices of K_{n-1}^+.
Therefore, the result follows. □

Theorem 4.3: Let G be a connected graph with k pendent vertices and $H_t = L(H_{t-1}^+)$, $t = 1, 2, \ldots$ Where $H_0 = G$ and $H_1 = L(G^+)$ then,

$$T[E_1(H_t)] = \sum_{i=1}^{k} (e_i + t)^2 \quad \text{and} \quad T[E_2(H_t)] = \sum_{1 \leq i < j \leq k} (e_i + t)(e_j + t) .$$

Proof: As G has k pendent vertices from Theorem 4.1, the graph H_t has k pendent vertices, $t = 1, 2, \ldots$

Therefore, $T[E_1(H_1)] = T[E_1(L(G^+))] = \sum_{i=1}^{k} (e_i + 1)^2$.

By induction, let $T[E_1(H_{t-1})] = T[E_1(L(H_{t-2}^+))] = \sum_{i=1}^{k} (e_i + t - 1)^2$.

Therefore $T[E_1(H_t)] = T[E_1(L(H_{t-1}^+))] = \sum_{i=1}^{k} [e_i + (t-1) - 1]^2$.

Hence, $T[E_1(H_t)] = \sum_{i=1}^{k} (e_i + t)^2$.

Similarly, $T[E_2(H_t)] = \sum_{1 \le i < j \le k} (e_i + t)(e_j + t)$. □

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