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SANDWICH THEOREMS FOR ANALYTIC FUNCTIONS DEFINED BY CERTAIN NEW OPERATORS

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Abstract: In this paper we obtain some subordination and superordination results involving certain new generalized operators for certain subclasses of analytic functions in the open unit disk. Our results improve previously known results.

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INTRODUCTION

Denote by U the open unit disc of the complex plane, $U = \{z \in C : |z| < 1\}$. Let $\mathrm{H}(U)$ be the space of analytic functions in U . For $n \in N, a \in C$ we define:

$$\mathrm{H}[a,n] = \{ f \in \mathrm{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\},\ z \in U,$$

$$A = \{ f \in \mathrm{H}(U) : f(z) = z + a_2 z^2 + a_3 z^3 + ...,\ \},\ z \in U.$$

For two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, the Hadamard product (or convolution) of f and g is defined by $(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z).$

Since we use the terms subordination and superordination, we review here those definition. Let $f, g \in \mathrm{H}(U)$. we say that the function f is subordinate to g , or the function g is superordinate to f , if there exists a Schwarz function w, analytic in U , with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$. In such a case we write $f \prec g$. Furthermore, if the function g is univalent in U , then we have the following equivalence (See [6] and [13]):

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\varphi : C^2 \times U \to C$ and $h(z)$ be univalent in U .If $p(z)$ is analytic in U and satisfies the first-order

differential subordination:

(1.1) $\varphi(p(z), zp'(z); z) \prec h(z),$

then $p(z)$ is a solution of the differential subordination (1.1). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.1) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants of (1.1) is called the best dominant. If $p(z)$ and $\varphi(p(z), zp'(z); z)$ are univalent functions in U and if $p(z)$ satisfies the first-order superordination

(1.2) $$h(z) \prec \varphi(p(z), zp'(z); z),$$

then $p(z)$ is called to be a solution of the differential superordination (1.2). A function $q \in \mathrm{H}(U)$ is called a subordinant of the solutions of the differential superordination (1.2) if $q(z) \prec p(z)$ for all the functions $p(z)$ satisfying (1.2). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all of the subordinants of (1.2) is said to be the best subordinant. Using the results of Miller and Mocanu [14], Bulboaca [5, 6] considered certain classes of first order differential superordinations. Ali et. al. [2], have used the results of Bulboaca [6] to obtain sufficient conditions for normalised analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1(z)$ and $q_2(z)$ are given univalent normalized functions in U.

A new generalized derivative of a function f is defined in [18] and is as follows:

DEFINITION 1.1: For $f \in A, m \in N_0 = N \cup \{0\}, \beta \geq 0$ and α a real number with $\alpha + \beta > 0,$ a new generalized multiplier transformation, denoted by $I^m_{\alpha,\beta}$, is defined by the following infinite series:

(1.3) $$I^m_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m a_k z^k ., z \in U.$$

It follows from (1.3) that

(1.4) $$(\alpha + \beta) I^{m+1}_{\alpha,\beta} f(z) = \alpha I^m_{\alpha,\beta} f(z) + \beta z (I^m_{\alpha,\beta} f(z))',$$

Remark 1.2 i) $I^m_{\alpha,1} f(z) = I^m_{\alpha} f(z), \alpha > -1,$ was defined in [8] and [9] (but considered for $\alpha \geq 0$) and $I^m_{l+1-\beta} f(z) = I^m_{l,\beta} f(z), l > -1, \beta \geq 0,$ was defined in [7] (but studied for $l \geq 0, \beta \geq 0$), ii) $I^m_{1,\beta} f(z) = D^m_{\beta} f(z), \quad \beta \geq 0$ was due to Al-Oboudi [1], iii) $D^m_1 f(z) = D^m f(z)$ was introduced by Salagean [17] and was considered for $m \geq 0$ in [4], and iv) $I^m_1 f(z)$ was investigated by Uralegaddi and Somanath [19].

We now define a new generalized integral operator $J^{m}_{\alpha,\beta} f(z),\ f(z) \in A$ as follows:

$$J^{0}_{\alpha,\beta} f(z) = f(z),$$

$$J^{1}_{\alpha,\beta} f(z) = J_{p,\alpha,\beta} f(z) = \left(\frac{\alpha+\beta}{\beta}\right) z^{1-\left(\frac{\alpha+\beta}{\beta}\right)} \int_0^z t^{\left(\frac{\alpha+\beta}{\beta}\right)-2} f(t)dt, z \in U,$$

$$J^{2}_{\alpha,\beta} f(z) = \left(\frac{\alpha+\beta}{\beta}\right) z^{1-\left(\frac{\alpha+\beta}{\beta}\right)} \int_0^z t^{\left(\frac{\alpha+\beta}{\beta}\right)-2} J^{1}_{p,\alpha,\beta} f(t)dt, z \in U,$$

$\ldots$

$$J^{m}_{\alpha,\beta} f(z) = \left(\frac{\alpha+\beta}{\beta}\right) z^{1-\left(\frac{\alpha+\beta}{\beta}\right)} \int_0^z t^{\left(\frac{\alpha+\beta}{\beta}\right)-2} J^{m-1}_{p,\alpha,\beta} f(t)dt$$

$$= J^{1}_{\alpha,\beta}\left(\frac{z}{1-z}\right) * J^{1}_{\alpha,\beta}\left(\frac{z}{1-z}\right) * \ldots * J^{1}_{\alpha,\beta}\left(\frac{z}{1-z}\right) * f(z)$$

$\leftarrow$ -------------------- m – times ------------------$\rightarrow$

where $m \in N_0 = N \bigcup \{0\}$, $\beta > 0$ and α a real number with $\alpha + \beta > 0$.

We see that for $f(z) \in A$, we have

(1.5) $$J^{m}_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha+\beta}{\alpha+k\beta}\right)^{m} a_k z^k, z \in U,$$

where $m \in N_0 = N \bigcup \{0\}$, $\beta > 0$ and α a real number with $\alpha + \beta > 0$.

From (1.5), it is easy to verify that

(1.6) $$(\alpha+\beta) J^{m}_{\alpha,\beta} f(z) = \alpha J^{m+1}_{\alpha,\beta} f(z) + \beta z (J^{m+1}_{\alpha,\beta} f(z))'.$$

Remark 1.3 i) $J^{m}_{1,1} f(z) = L^{m}_{1} f(z) = L^{m} f(z)$ (See [11, 12]) ii) $J^{m}_{1-\beta,\beta} f(z) = L^{m}(\beta) f(z), \beta > 0$ (See [15]) and iii) $J^{m}_{\alpha,1} f(z) = J^{m}_{\alpha} f(z), \alpha > -1.$

In this paper we will determine some properties on admissible functions defined with a new generalized differential operator and also with a new generalized integral operator.

PRELIMINARIES

In order to prove our results, we need the following definition and lemmas.

Definition 2.1 ([14]) we denote by Q, the set of all functions q that are analytic and injective on $\overline{U}\setminus E(q)$, where $E(q)=\{\zeta\in\partial U:\lim_{z\to\zeta}q(z)=\infty\}$ and are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(q)$.

Lemma 2.2([10]**)** Let $q(z)$ be univalent in U, $\gamma\in C^*=C\setminus\{0\}$ and suppose that

$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\mathrm{Re}\left(\frac{1}{\gamma}\right)\right\}, z\in U.$ If $p(z)$ is analytic in U, with $p(0)=q(0)$ and

$p(z)+\gamma zp'(z)\prec q(z)+\gamma zq'(z)$, then $p(z)\prec q(z)$, and $q(z)$ is the best dominant.

Lemma 2.3 ([5]**)** Let $q(z)$ be convex in U with $q(0)=a$ and $\gamma\in C,\ \mathrm{Re}(\gamma)>0.$ If $p(z)\in \mathrm{H}[a,1]$ and $p(z)+\gamma zp'(z)$ is univalent in U, then $q(z)+\gamma zq'(z)\prec p(z)+\gamma zp'(z)$, implies

$q(z)\prec p(z)$ and $q(z)$ is the best subordinant.

MAIN RESULTS

Unless otherwise mentioned, we shall assume in the remainder of the paper that $m\in N_0=N\cup\{0\}, z\in U$ and the powers are understood as principle values.

Theorem 3.1 Let $f\in A,\mu>0,\lambda\in C^*,\beta>0,\alpha$ a real number such that $\alpha+\beta>0.$ Let the function q be univalent in U and suppose that it satisfies the condition

$$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\mathrm{Re}\left(\frac{\mu}{\lambda}\right)\right\}.$$

Let

(3.1) $$\Phi(m,\mu,\lambda,\alpha,\beta;z)=\left(1-\lambda\left(\frac{\alpha+\beta}{\beta}\right)\right)\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}+\lambda\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}\left(\frac{I^{m+1}_{\alpha,\beta}f(z)}{I^m_{\alpha,\beta}f(z)}\right).$$

If

(3.2) $$\Phi(m,\mu,\lambda,\alpha,\beta;z)\prec q(z)+\frac{\lambda}{\mu}zq'(z),$$

then

$$\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}\prec q(z),$$ and $q(z)$ is the best dominant.

Proof. We define the function

(3.3) $$p(z)=\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}.$$

By finding the logarithmic derivative of (3.3) and using the identity (1.4), we obtain

(3.4) $$p(z)+\frac{\lambda}{\mu}zp'(z)=\left(1-\lambda\left(\frac{\alpha+\beta}{\beta}\right)\right)\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}+\lambda\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}\left(\frac{I^{m+1}_{\alpha,\beta}f(z)}{I^m_{\alpha,\beta}f(z)}\right).$$

From (3.1), (3.2) and (3.4), we get

$$p(z)+\frac{\lambda}{\mu}zp'(z)\prec q(z)+\frac{\lambda}{\mu}zq'(z).$$

We apply now Lemma 2.2 with $\gamma=\frac{\lambda}{\mu}$ to obtain the conclusion of our theorem.

Remark 3.2 i) Taking $\alpha=l+1-\beta$ in Theorem 3.1, we obtain Theorem 1 of Aouf et.al. [3] (Considered for $l\geq 0$).but our result hold true for $l>-1$. ii) Putting $\alpha=1-\beta$ in Theorem 3.1, we get Theorem 3.1 of Raducanu et.al. [16].

For $\beta=1$ in Theorem 3.1, we get the following corollary.

Corollary 3.3 Let $f\in A,\mu>0,\lambda\in C^*,\alpha$ a real number such that $\alpha>-1$. Let the function q be univalent in U and suppose that it satisfies the condition

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\operatorname{Re}\left(\frac{\mu}{\lambda}\right)\right\}.$$

Let

(3.5) $$\Phi_1(m,\mu,\lambda,\alpha;z)=(1-\lambda(\alpha+1))\left(\frac{I^m_{\alpha}f(z)}{z}\right)^{\mu}+\lambda(\alpha+1)\left(\frac{I^m_{\alpha}f(z)}{z}\right)^{\mu}\left(\frac{I^{m+1}_{\alpha}f(z)}{I^m_{\alpha}f(z)}\right).$$

If

$$\Phi_1(m,\mu,\lambda,\alpha;z)\prec q(z)+\frac{\lambda}{\mu}zq'(z),$$

then

$$\left(\frac{I^m_{\alpha}f(z)}{z}\right)^{\mu}\prec q(z)$$, and $q(z)$ is the best dominant.

We obtain the following result from Theorem 3.1, by taking $m=0$.

Corollary 3.4 Let $f\in A,\mu>0,\lambda\in C^*,\beta\geq 0,\alpha$ a real number such that $\alpha+\beta>0$. Let the function q be univalent in U and suppose that it satisfies the condition

$$\operatorname{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\operatorname{Re}\left(\frac{\mu}{\lambda}\right)\right\}.$$

Let

(3.6) $$\Phi_2(\mu,\lambda,\alpha,\beta;z)=\left(1-\lambda\left(\frac{\alpha+\beta}{\beta}\right)\right)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{f(z)}{z}\right)^{\mu}\left(\frac{\alpha f(z)+\beta zf'(z)}{(\alpha+\beta)f(z)}\right).$$

If

$$\Phi_2(\mu,\lambda,\alpha,\beta;z) \prec q(z)+\frac{\lambda}{\mu}zq'(z),$$

then

$$\left(\frac{f(z)}{z}\right)^{\mu} \prec q(z)$$, and $q(z)$ is the best dominant.

We consider a particular convex function $q(z)=\frac{1+Az}{1+Bz}$, to give the following application of Theorem 3.1.

Corollary 3.5 Let $A,B,\in C, A\neq B$ such that $|B|<1, \mu>0, \lambda\in C^*, \beta>0, \alpha$ a real number such that $\alpha+\beta>0$ and suppose that $\mathrm{Re}\left[\frac{1-Bz}{1+Bz}\right]>\max\left\{0,-\mathrm{Re}\left(\frac{\mu}{\lambda}\right)\right\}$. If $f(z)\in A$ satisfies the condition

$$\Phi(m,\mu,\lambda,\alpha,\beta;z) \prec \frac{1+Az}{1+Bz}+\frac{\lambda}{\mu}\frac{(A-B)z}{(1+Bz)^2},$$

where $\Phi(m,\mu,\lambda,\alpha,\beta;z)$ is given by (3.1), then

$$\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu} \prec \frac{1+Az}{1+Bz}$$, and $\frac{1+Az}{1+Bz}$ is the best dominant.

In a manner similar to that of Theorem 3.1, we can easily prove the following theorem, using the identity (1.6).

Theorem 3.6 Let $f\in A, \mu>0, \lambda\in C^*, \beta>0, \alpha$ a real number such that $\alpha+\beta>0$. Let the function q be univalent in U and suppose that it satisfies the condition

$$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\mathrm{Re}\left(\frac{\mu}{\lambda}\right)\right\}.$$

Let

(3.7) $$\Psi(m,\mu,\lambda,\alpha,\beta;z)=\left(1-\lambda\left(\frac{\alpha+\beta}{\beta}\right)\right)\left(\frac{J^{m+1}_{\alpha,\beta}f(z)}{z}\right)^{\mu}+\lambda\left(\frac{\alpha+\beta}{\beta}\right)\left(\frac{J^{m+1}_{\alpha,\beta}f(z)}{z}\right)^{\mu}\left(\frac{J^{m}_{\alpha,\beta}f(z)}{J^{m+1}_{\alpha,\beta}f(z)}\right).$$

If

$$\Psi(m,\mu,\lambda,\alpha,\beta;z) \prec q(z)+\frac{\lambda}{\mu}zq'(z),$$

then

$$\left(\frac{J^{m+1}_{\alpha,\beta}f(z)}{z}\right)^{\mu} \prec q(z)$$, and $q(z)$ is the best dominant.

For $\beta=1$ in Theorem 3.6, we get the following corollary.

Corollary 3.7 Let $f\in A, \mu>0, \lambda\in C^*, \alpha$ a real number such that $\alpha>-1$. Let the function q be univalent in U and suppose that it satisfies the condition

$$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}\right]>\max\left\{0,-\mathrm{Re}\left(\frac{\mu}{\lambda}\right)\right\}.$$

Let

(3.8) $$\Psi_1(m,\mu,\lambda,\alpha;z)=(1-\lambda(\alpha+1))\left(\frac{J_\alpha^{m+1}f(z)}{z}\right)^\mu+\lambda(\alpha+1)\left(\frac{J_\alpha^{m+1}f(z)}{z}\right)^\mu\left(\frac{J_\alpha^{m}f(z)}{J_\alpha^{m+1}f(z)}\right).$$

If

$$\Psi_1(m,\mu,\lambda,\alpha;z)\prec q(z)+\frac{\lambda}{\mu}zq'(z),$$

then

$$\left(\frac{J_\alpha^{m+1}f(z)}{z}\right)^\mu\prec q(z)$$, and $q(z)$ is the best dominant.

The next theorem is a result concerning a differential superordination.

Theorem 3.8 Let q be convex in U with $q(0)=1,\mu>0,\lambda\in C$ with $\operatorname{Re}(\lambda)>0,\beta>0,$ and α a real number such that $\alpha+\beta>0$. If $f(z)\in A$ such that $\left(\frac{I_{\alpha,\beta}^{m}f(z)}{z}\right)^\mu\in \mathrm{H}[1,1]\cap \mathrm{Q}$, $\Phi(m,\mu,\lambda,\alpha,\beta;z)$ is univalent in U and satisfies the superordination

(3.9) $$q(z)+\frac{\lambda}{\mu}zq'(z)\prec\Phi(m,\mu,\lambda,\alpha,\beta;z),$$

where $\Phi(m,\mu,\lambda,\alpha,\beta;z)$ is given by (3.1), then

$$q(z)\prec\left(\frac{I_{\alpha,\beta}^{m}f(z)}{z}\right)^\mu$$, and $q(z)$ is the best subordinant.

Proof. Let $p(z)$ be given by (3.3) and proceeding as in the proof of Theorem 3.1, (3.9) becomes

$$q(z)+\frac{\lambda}{\mu}zq'(z)\prec p(z)+\frac{\lambda}{\mu}zp'(z).$$

The proof follows by an application of Lemma 2.3.

We get the following corollary on putting $\beta=1$ in Theorem 3.8.

Corollary 3.9 Let q be convex in U with $q(0)=1,\mu>0,\lambda\in C$ with $\operatorname{Re}(\lambda)>0,$ and α a real number such that $\alpha>-1$ If $f(z)\in A$ such that $\left(\frac{I_{\alpha}^{m}f(z)}{z}\right)^\mu\in \mathrm{H}[1,1]\cap \mathrm{Q}$, $\Phi_1(m,\mu,\lambda,\alpha;z)$ is univalent in U and satisfies the superordination $q(z)+\frac{\lambda}{\mu}zq'(z)\prec\Phi_1(m,\mu,\lambda,\alpha;z)$, where $\Phi_1(m,\mu,\lambda,\alpha;z)$ is given by (3.5), then $q(z)\prec\left(\frac{I_{\alpha}^{m}f(z)}{z}\right)^\mu$, and $q(z)$ is the best subordinant.

We obtain the following result from Theorem 3.8, by putting $m=0$.

Corollary 3.10 Let the function q be convex in U with $q(0)=1, \mu>0, \lambda\in C^*, \beta\geq 0, \alpha$ a real number such that $\alpha+\beta>0$. If $f(z)\in A$ such that $\left(\frac{I_\alpha^m f(z)}{z}\right)^\mu \in \mathrm{H}[1,1]\cap \mathrm{Q}$, $\Phi_2(m,\mu,\lambda,\alpha;z)$ is univalent in U and satisfies the superordination $\Phi_2(\mu,\lambda,\alpha,\beta;z)\prec q(z)+\frac{\lambda}{\mu}zq'(z)$, where $\Phi_2(m,\mu,\lambda,\alpha;z)$ is given by (3.6), then $q(z)\prec\left(\frac{f(z)}{z}\right)^\mu$, and $q(z)$ is the best subordinant.

In a manner similar to that of Theorem 3.8, we can easily prove the following theorem.

Theorem 3.10 Let q be convex in U with $q(0)=1, \mu>0, \lambda\in C$ with $\mathrm{Re}(\lambda)>0, \beta>0,$ and α a real number such that $\alpha+\beta>0$. If $f(z)\in A$ such that $\left(\frac{J_{\alpha,\beta}^{m+1} f(z)}{z}\right)^\mu \in \mathrm{H}[1,1]\cap \mathrm{Q}$, $\Psi(m,\mu,\lambda,\alpha,\beta;z)$ is univalent in U and satisfies the superordination

$$q(z)+\frac{\lambda}{\mu}zq'(z)\prec\Psi(m,\mu,\lambda,\alpha,\beta;z),$$

where $\Psi(m,\mu,\lambda,\alpha,\beta;z)$ is given by (3.7), then

$$q(z)\prec\left(\frac{J_{\alpha,\beta}^{m+1} f(z)}{z}\right)^\mu, \text{ and } q(z) \text{ is the best subordinant.}$$

Taking $\beta=1$ in Theorem 3.10, we get the following corollary.

Corollary 3.11 Let q be convex in U with $q(0)=1, \mu>0, \lambda\in C$ with $\mathrm{Re}(\lambda)>0,$ and α a real number such that $\alpha>-1$ If $f(z)\in A$ such that $\left(\frac{J_{\alpha}^{m+1} f(z)}{z}\right)^\mu \in \mathrm{H}[1,1]\cap \mathrm{Q}$, $\Psi_1(m,\mu,\lambda,\alpha;z)$ is univalent in U and satisfies the superordination $q(z)+\frac{\lambda}{\mu}zq'(z)\prec\Psi_1(m,\mu,\lambda,\alpha;z)$, where $\Psi_1(m,\mu,\lambda,\alpha;z)$ is given by (3.8), then $q(z)\prec\left(\frac{J_{\alpha}^{m+1} f(z)}{z}\right)^\mu$, and $q(z)$ is the best subordinant.

Combining the results of Theorem 3.1 and Theorem 3.8, we state the following sandwich result.

Theorem 3.12 Let q_1 and q_2 be convex in U with $q_1(0)=q_2(0)=1, \mu>0, \lambda\in C$ with $\mathrm{Re}(\lambda)>0$, $\beta>0,$ and α a real number such that $\alpha+\beta>0.$ If $f(z)\in A$ such that $\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}\in \mathrm{H}[1,1]\cap \mathrm{Q},$

$\Phi(m,\mu,\lambda,\alpha,\beta;z)$ is univalent in U and satisfies the superordination

$$q_1(z)+\frac{\lambda}{\mu}zq_1^{'}(z)\prec\Phi(m,\mu,\lambda,\alpha,\beta;z)\prec q_2(z)+\frac{\lambda}{\mu}zq_2^{'}(z)$$

where $\Phi(m,\mu,\lambda,\alpha,\beta;z)$ is given by (3.1), then $q_1(z)\prec\left(\frac{I^m_{\alpha,\beta}f(z)}{z}\right)^{\mu}\prec q_2(z)$, $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

Combining the results of Theorem 3.6 and Theorem 3.10, we obtain the following sandwich result.

Theorem 3.13 Let q_1 and q_2 be convex in U with $q_1(0)=q_2(0)=1, \mu>0, \lambda\in C$ with $\mathrm{Re}(\lambda)>0$, $\beta>0,$ and α a real number such that $\alpha+\beta>0.$ If $f(z)\in A$ such that $\left(\frac{J^{m+1}_{\alpha,\beta}f(z)}{z}\right)^{\mu}\in \mathrm{H}[1,1]\cap \mathrm{Q},$ $\Psi(m,\mu,\lambda,\alpha,\beta;z)$ is univalent in U and satisfies the superordination

$$q_1(z)+\frac{\lambda}{\mu}zq_1^{'}(z)\prec\Psi(m,\mu,\lambda,\alpha,\beta;z)\prec q_2(z)+\frac{\lambda}{\mu}zq_2^{'}(z)$$

where $\Psi(m,\mu,\lambda,\alpha,\beta;z)$ is given by (3.7), then $q_1(z)\prec\left(\frac{J^{m+1}_{\alpha,\beta}f(z)}{z}\right)^{\mu}\prec q_2(z)$, $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively

Remark 3.14 Combining Corollaries 3.3, 3.9 and 3.7, 3.11, we get the corresponding sandwich results for the operators I^m_{α} and J^{m+1}_{α}, respectively.

Remark 3.15 Putting $\alpha=l+1-\beta$ Theorem 3.8 and Theorem 3.12, we obtain Theorem 3 and Theorem 5, respectively, of Aouf et.al. [3] (Considered for $l\geq 0$). But our results hold true for $l>-1.$

Remark 3.16 For $\alpha=1-\beta$ in Theorem 3.8 and Theorem 3.12, we get Theorem 3.6 and Theorem 3.9, respectively, of Raducanu et.al. [16].

Remark 3.17 Taking $\alpha=1-\beta$ in Corollary 3.4 and Corollary 3.5, we obtain Corollary 3.2 and Corollary 3.5, respectively, of Raducanu et.al. [16].

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