Volume 1, No. 1, January 2013
Journal of Global Research in Mathematical Archives
RESEARCH PAPER
Available Online at http:/ /www.jgrma.info

Adjacency Algebra of Unitary Cayley Graph

A. Satyanarayana Reddy
Department of Mathematics,
Shiv Nadar University,
Dadri, India-203207
Email:satya8118@gmail.com

Abstract

A few properties of unitary Cayley graphs are explored using their eigenvalues. It is shown that the adjacency algebra of a unitary Cayley graph is a coherent algebra. Finally, a class of unitary Cayley graphs that are distance regular are also obtained.

Key Words: Adjacency Algebra, Circulant Graph, Coherent Algebra, Distance Regular Graph, Ramanujan's sum .
AMS(2010): 05C25, 05C50

1 Introduction and Preliminaries

Fix a positive integer n and let $\mathbb{M}_n(\mathbb{C})$ denote the algebra of all $n \times n$ matrices over $\mathbb{C}$, the set of complex numbers. Let X be a simple graph/digraph on n vertices. Then the adjacency matrix of X, denoted by $A(X) = [a_{ij}]$ (or simply A), is an $n \times n$ matrix, where a_{ij} equals 1 when the vertices i and j are adjacent ($\{i,j\}/(i,j)$ is an edge/directed edge) in X and 0, otherwise. The *adjacency algebra* of X, denoted $\mathcal{A}(X)$, is a subalgebra of $\mathbb{M}_n(\mathbb{C})$ and it consists of all polynomials in A with coefficients from $\mathbb{C}$.

For any two vertices u and v of a connected graph X, let $d(u,v)$ denote the length of the shortest path from u to v. Then the *diameter* of a connected graph $X = (V,E)$ is $\max\{d(u,v) : \ u,v \in V\}$. It is shown in Biggs [2] that if X is a connected graph with diameter D, then $D+1 \leq \dim(\mathcal{A}(X)) \leq n$, where $\dim(\mathcal{A}(X))$ is the dimension of $\mathcal{A}(X)$ as a vector space over $\mathbb{C}$. We now state the following two results associated with connected regular graphs.

Lemma 1.1 (A. J. Hoffman [8])**.** *A graph X is a connected regular graph if and only if $\mathbf{J} \in \mathcal{A}(X)$, where $\mathbf{J}$ is the matrix of all 1's.*

A connected graph is a *distance regular* if for any two vertices u and v, the number of vertices at distance i from u and j from v depends only on i, j, and the distance between u and v. By definition, these graphs are regular. A distance regular graph with diameter 2 is called *strongly regular graph.*

Lemma 1.2 (S. S. Shrikhande and Bhagwandas [15])**.** *A regular connected graph X is strongly regular if and only if it has exactly three distinct eigenvalues.*

The graph X is *distance transitive* if for all vertices u,v,x,y of X such that $d(u,v) = d(x,y)$, then there is a g in $Aut(X)$ (the automorphism group of graph X) satisfying $g(u) = x$ and $g(v) = y$.

Every distance transitive graph is distance regular. To know more about distance regular and distance transitive graphs refer A. E. Brouwer, A. M. Cohen, A. Neumaier [3].

For two matrices $A, B \in \mathbb{M}_n(\mathbb{C})$, their *Hadamard product*, denoted $A \circ B$, is also an $n \times n$ matrix with $(A \circ B)_{ij} = A_{ij}B_{ij}$ for $1 \leq i, j \leq n$. Two matrices A and B are said to be *disjoint* if their Hadamard product is the zero matrix.

Theorem 1.3. *[3] Let $\mathcal{M}$ be a vector space of symmetric $n \times n$ matrices. Then $\mathcal{M}$ has a basis of mutually disjoint $0, 1$-matrices if and only if $\mathcal{M}$ is closed under Hadamard multiplication.*

A subalgebra of $\mathbb{M}_n(\mathbb{C})$ containing $I, \mathbf{J}$, where I is the identity matrix, is said to be a *coherent algebra* if it is closed under Hadamard product and conjugate transposition. For example, $M_n(\mathbb{C})$ and $\mathcal{A}(K_n)$, the adjacency algebra of complete graph K_n are the largest and smallest coherent algebras respectively. Now we will see an example of a non-trivial coherent algebra.

A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be *circulant* if $a_{ij} = a_{1,\, j-i+1\ (mod\ n)}$, whenever $2 \leq i \leq n$ and $1 \leq j \leq n$. From the definition, it is clear that if A is circulant, then for each $i \geq 2$ the elements of the i-th row are obtained by cyclically shifting the elements of the $(i-1)$-th row one position to the right. So it is sufficient to specify its first row. Let W_n be a circulant matrix of order n with $[0\ 1\ 0 \ldots 0]$ as its first row . Then the following result of Davis [6] establishes that every circulant matrix of order n is a polynomial in W_n.

Lemma 1.4. *[6] Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A is circulant if and only if it is a polynomial over $\mathbb{C}$ in W_n.*

Let DC_n denotes the *directed cycle* with n vertices. Then it is easy to see that W_n is the adjacency matrix of DC_n and $\mathcal{A}(DC_n)$ is a coherent algebra of dimension n. Further, $\{W_n^0, W_n^1, W_n^2, \ldots, W_n^{n-1}\}$ is the unique basis of $\mathcal{A}(DC_n)$ with mutually disjoint $0, 1$-matrices.

Let X be a graph and A be its adjacency matrix. The *coherent closure* of X, denoted by $\mathcal{CC}(X)$, is the smallest coherent algebra containing A. A graph X is said to be *pattern polynomial graph* if $\mathcal{A}(X) = \mathcal{CC}(X)$. For example, distance regular graphs are pattern polynomial graphs. In particular, let C_n be the *cycle graph* with n vertices. Then $A(C_n) = W_n + W_n^{n-1}$, the adjacency matrix of C_n. Let $D_i = W_n^i + W_n^{n-i}$ for $1 \leq i < \lfloor \frac{n}{2} \rfloor$. For $\tau = \lfloor \frac{n}{2} \rfloor$,

$$D_\tau = \begin{cases} W_n^\tau & \text{if } n \textit{ is even}, \\ W_n^\tau + W_n^{n-\tau}, & \text{if } n \textit{ is odd}. \end{cases}$$

The identity

$$(x^k + x^{-k}) = (x + x^{-1})(x^{k-1} + x^{1-k}) - (x^{k-2} + x^{2-k})$$

enables us to establish readily by mathematical induction that $x^k + x^{-k}$ is a monic polynomial in $x + x^{-1}$ of degree k with integral coefficients. Consequently, D_i's for $1 \leq i \leq \tau$ are polynomials of degree $\leq i$ in $D_1 = A(C_n)$ over $\mathbb{C}$. Hence $\{D_0 = I, D_1, \ldots, D_\tau\}$ is the unique basis for $\mathcal{A}(C_n)$ with mutually disjoint $0, 1$-matrices. Thus from Theorem 1.3, C_n is a pattern polynomial graph. For more examples of pattern polynomial graphs refer [14].

For a fixed positive integer n, let $\mathbb{Z}_n$ denote the set of integers modulo n. It is well known that $\mathbb{Z}_n$ forms a cyclic group with respect to addition modulo n and $U_n = \{k : 1 \leq k \leq n, \gcd(k, n) = 1\} \subset \mathbb{Z}_n$ forms a group with respect to multiplication modulo n. Also, for any set S, if $|S|$ denotes the cardinality of the set S, then $|U_n| = \varphi(n)$, the famous *Euler-totient function.* Let $\zeta_n \in \mathbb{C}$ denote a *primitive n-th root of unity, i.e.,* $(\zeta_n)^n = 1$ and $(\zeta_n)^k \neq 1$ for any $k = 1, 2, \ldots, n-1$. Then it is well known that the multiplicative group generated by ζ_n is isomorphic to the additive group $\mathbb{Z}_n$ and the set $\{(\zeta_n)^k : k \in U_n\}$ is the collection of all primitive n-th roots of unity.

For any divisor d of n, let X_d^n (or in short X_d, when the positive integer n is clear from the context) denote the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, U_d)$ that consists of the elements of $\mathbb{Z}_n$ as vertices and two vertices $x, y \in \mathbb{Z}_n$ are adjacent (or $\{x, y\}$ is an edge) in X_d^n if $x - y(mod\, n) \in U_d$. A graph is called *circulant* if its adjacency matrix is a circulant matrix. Since $\mathbb{Z}_n$ is a cyclic group, for each divisor d of n the Cayley graph X_d is a circulant graph. The graph X_n is commonly known as the *unitary Cayley graph.*

In the remaining part of this section, we provide a few results, required for this paper. In the Section 2, we give few properties of unitary Cayley graphs which are derived from their eigenvalues. In Section 3, we show that every unitary Cayley graph is a pattern polynomial graph. We also found the values of n for which X_n is a distance regular and strongly regular graphs.

Recall that a positive integer is said to be *square free* if its decomposition into prime numbers does not have any repeated factors. Let $\gamma(n) = \prod\limits_{p|n} p$ be the square free part of n, where p is a prime number.

Lemma 1.5 ([7]). *Let n be a positive integer. Then $\sum\limits_{k \in U_n} \zeta_n^k = \mu(n)$, where*

$$\mu(n) = \begin{cases} 0, & \text{if } n \text{ is not square free,} \\ 1, & \text{if } n \text{ has even number of prime factors,} \\ -1, & \text{if } n \text{ has odd number of prime factors.} \end{cases}$$

Now we will give two results on Ramanujan's sum.

Definition 1.6. *For any positive integer n and non-negative integer m, the* Ramanujan's sum, *is defined as $c_n(m) = \sum\limits_{k \in U_n} (\zeta_n^k)^m$.singular/non-singular*

For example $c_n(0) = \varphi(n)$ and from Lemma 1.5 we have $c_n(1) = \mu(n)$.

Lemma 1.7. *[1, 11, 12] Fix positive integers m and n and let $\mu(n)$ and $c_n(m)$ be as defined earlier. Then*

1. *for each divisor d of n, $c_n(d) = \mu(\frac{n}{d})\frac{\varphi(n)}{\varphi(\frac{n}{d})}$. Furthermore $c_n(m) = c_n(d)$ for all m for which $\gcd(m, n) = d$.*
2. *$c_n(m) = \sum_{d|\gcd(n,m)} \mu(n/d)d$. In particular we have $c_n(m) \in \mathbb{Z}$.*

Now we consider a polynomial [Motose [12], Laszlo Toth [18]] with coefficients as Ramanujan's sums, namely, $R_n(x) = c_n(0) + c_n(1)x + c_n(2)x^2 + \cdots + c_n(n-1)x^{n-1}$. Then the following theorem due to Laszlo Toth [18] provides a few properties of the polynomial $R_n(x)$.

Theorem 1.8. *[18] Let $n \geq 1$.*

1. *The number of non-zero coefficients of $R_n(x)$ is $\gamma(n)$ and the degree of $R_n(x)$ is $n - \frac{n}{\gamma(n)}$.*
2. *$R_n(x)$ has coefficients ± 1 if and only if n is square free and in this case the number of coefficients ± 1 of $R_n(x)$ is $\varphi(n)$ for n is odd and is $2\varphi(n/2)$ for n is even.*

2 Few properties of unitary Cayley graphs from their eigenvalues

In this section, we provide few properties of unitary Cayley graphs using their eigenvalues. Most of these results are stated in [10, 17]. There are two tables consisting of eigenvalues, minimal polynomial and characteristic polynomial of adjacency matrix of X_n.

Let $A \in M_n(\mathbb{C})$ be a circulant matrix. Then from Lemma 1.4, there exists a unique polynomial $p_A(x) \in \mathbb{C}[x]$ of degree $\leq n-1$ such that $A = p_A(W_n)$. We call $p_A(x)$, the representer polynomial of A. Then the following result about circulant matrices is well known.

Lemma 2.1. *Let $A \in M_n(\mathbb{C})$ be a circulant matrix with $[a_0 \ a_1 \dots a_{n-1}]$ as its first row. Then $p_A(x) = \sum_{i=0}^{n-1} a_i x^i \in \mathbb{C}[x]$ and the eigenvalues of A are given by $p_A(\zeta_n^k)$ for $k = 0, 1, \dots, n-1$. Further the matrix A is singular if and only if* $\deg(\gcd(p_A(x), x^n - 1)) > 1$.

Let us denote the adjacency matrix of X_d by A_d. Then from the definition of X_d, $A_d = \sum_{k \in U_d} W_n^{\frac{nk}{d}}$. Hence its representer polynomial is $p_{A_d}(x) = \sum_{k \in U_d} x^{\frac{nk}{d}}$. Note the polynomial $p_{A_n}(x)$ is same as the polynomial $\Psi_n(x)$ defined in the paper L. Toth [18]. Further, he proved that $\Phi_n(x)$ divides $\Psi_n(x) - \mu(n)$. Hence we have the following result.

Lemma 2.2. *Let n be a positive integer. Then $\Phi_n(x) | p_{A_n}(x) - \mu(n)$.*

If n is not square free number then, from Lemma 1.5, $\mu(n) = 0$. Hence from the Lemma 2.1,2.2, we have the following result. And its converse is also true, and which is shown as a part in a corollary to next theorem. Recall a graph is said to be singular/non-singular if its adjacency matrix is singular/non-singular.

Corollary 2.3. *Let n be a positive integer. Then X_n is non-singular graph if and only if n is square free.*

Theorem 2.4. *[10, 17] Let λ_i $(0 \leq i \leq n-1)$ are the eigenvalues of X_n. Then $\lambda_i = c_n(i)$ $(0 \leq i \leq n-1)$.*

By fixing the following notations, we will see few applications of above theorem. Let $n = 2^{c_0} p_1^{c_1} p_2^{c_2} \dots p_r^{c_r}$ where $p_1 < p_2 \dots < p_r$ are distinct odd primes.
Let us denote $D^* = \{a_{i_1} a_{i_2} \cdots a_{i_t} | i_1, i_2, \dots, i_t \in \{1, 2, \dots, r\}\}$ where $a_j = p_j - 1 \ for \ 1 \leq j \leq r$. Note for each element $b \in D^*$ there is a number t $(1 \leq t \leq r)$ associated with it. By using these notations and the results Lemma 1.7, Theorem 1.8 and Theorem 2.4 we have the following tables. Recall, spectrum of a graph X is denoted by $\sigma(X) = \begin{pmatrix} \lambda_1 & \dots & \lambda_k \\ m_1 & \dots & m_k \end{pmatrix}$, where $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of X and $m_1, \dots, m_k$ are their corresponding multiplicities respectively.

Table 1: Spectrum of unitary Cayley Graphs

n	$\sigma(X_n)$(Spectrum of the graph)
p, p is prime	$\begin{pmatrix} -1 & p-1 \\ p-1 & 1 \end{pmatrix}$
p^k, p is prime and $k>1$	$\begin{pmatrix} -p^{k-1} & 0 & (p-1)p^{k-1} \\ p-1 & p^k-p & 1 \end{pmatrix}$
$2p$, p is prime	$\begin{pmatrix} -(p-1) & -1 & 1 & p-1 \\ 1 & p-1 & p-1 & 1 \end{pmatrix}$
pq, where p and q are odd primes	$\begin{pmatrix} 1 & -(p-1) & -(q-1) & \varphi(pq) \\ \varphi(pq) & q-1 & p-1 & 1 \end{pmatrix}$
square free even number	$\begin{pmatrix} -1 & 1 & b & -b \\ \varphi(n) & \varphi(n) & \varphi(n)/b & \varphi(n)/b \end{pmatrix} \forall b \in D^*$
square free odd number	$\begin{pmatrix} (-1)^r & [(-1)^{r+t}b] \\ \varphi(n) & \varphi(n)/b \end{pmatrix} \forall b \in D^*$
even but not square free	$\begin{pmatrix} 0 & -n/\gamma(n) & n/\gamma(n) & (n/\gamma(n))b & -(n/\gamma(n))b \\ n-\gamma(n) & \varphi(n) & \varphi(n) & \varphi(n)/b & \varphi(n)/b \end{pmatrix} \forall b \in D^*$
odd but not square free	$\begin{pmatrix} 0 & (-1)^r(n/\gamma(n)) & [(-1)^{r+t}b](n/\gamma(n)) \\ n-\gamma(n) & \varphi(n) & \varphi(n)/b \end{pmatrix} \forall b \in D^*$

Table 2: Characteristic and minimal polynomials of unitary Cayley graphs

n	Minimal polynomial	Characteristic polynomial
p, p is prime	$(x+1)(x-(p-1))$	$(x+1)^{p-1}(x-(p-1))$
p^k p is prime and $k>1$	$x(x-(p-1)p^{k-1})(x+p^{k-1})$	$x^{p^k-p}(x-(p-1)p^{k-1})(x+p^{k-1})^{p-1}$
square free and even	$(x^2-1)\prod_{b\in D^*}(x^2-b^2)$	$(x^2-1)^{\varphi(n)}\prod_{b\in D^*}(x^2-b^2)^{\varphi(n)/b}$
square free and odd	$(x-(-1)^r)\prod_{b\in D^*}(x-(-1)^{r+t}b)$	$(x-(-1)^r)^{\varphi(n)}\prod_{b\in D^*}(x-(-1)^{r+t}b)^{\varphi(n)/b}$
not square free, even and $r>1$	$x(x^2-(n/\gamma(n))^2)\cdot \prod_{b\in D^*}(x^2-([n/\gamma(n)]b)^2)$	$x^{n-\gamma(n)}(x^2-(n/\gamma(n))^2)\cdot \prod_{b\in D^*}(x^2-([n/\gamma(n)]b)^2)^{\varphi(\gamma(n))/b}$
not square free, odd and $r>1$	$x(x-(-1)^r(n/\gamma(n)))\cdot \prod_{b\in D^*}(x-(-1)^{r+t}([n/\gamma(n)]b))$	$x^{n-\gamma(n)}(x^2-(n/\gamma(n))^2)\cdot \prod_{b\in D^*}(x-(-1)^{r+t}([n/\gamma(n)]b))^{\varphi(\gamma(n))/b}$

As a consequence of Theorems 1.8, 2.4, Lemma 1.7 and above tables, we have the following results. One can also refer the book by Dragos M. Cvetkovic, Michael Doob & Horst Sachs [5] for further clarification.

Corollary 2.5. *Let n be a positive integer.*

1. *X_n is non-singular if and only if n is square free.*
2. *[10] a)The number of non-zero eigenvalues of X_n are $\gamma(n)$ or the nullity (Dimension of the null space of A_n) of X_n is $n-\gamma(n)$.*

b) X_n *is an integral graph for every* n*. Further every non-zero eigenvalue of* X_n *is a divisor of* $\varphi(n)$.
c)If n *is not square free, then none of the eigenvalues of* X_n *is* 1 *or* -1.

3. *Let* $\tau(n)$ *is the number of positive divisors of* n*. Then the degree of minimal polynomial of* X_n *is*

$$\begin{cases} \tau(\gamma(n))+1, & \textit{if } n \textit{ is not square free,} \\ \tau(n), & \textit{if } n \textit{ is square free.} \end{cases}$$

4. *The*

$$\det(A(X_n)) = \begin{cases} 0, & \textit{if } n \textit{ is not square free,} \\ -1, & \textit{if } n\textit{=2,} \\ p-1, & \textit{if } n=p \textit{ is an odd prime,} \\ -(p-1)^2, & \textit{if } n=2p \textit{ where } p \textit{ is an odd prime,} \\ (p-1)^q(q-1)^p, & \textit{if } n=pq \textit{ where } p \textit{ ans } q \textit{ are odd primes,} \\ (-1)^r \prod\limits_{b\in D^*}[(-1)^t b]^{\varphi(n)/b}, & \textit{if } n \textit{ is a square free odd number,} \\ \prod\limits_{b\in D^*}(-1)^{\varphi(n)/b} b^{(2\varphi(n))/b}, & \textit{if } n \textit{ is a square free even number.} \end{cases}$$

The following result gives the graph theoretic properties of X_n.

Corollary 2.6. *Let* n *be a positive integer.*

1. *[10]* X_n *bipartite graph if and only if* n *even number. Further* X_n *is complete bipartite graph if and only if* $n = 2^k$ *for some* $k \geq 1$.
   X_n *is complete graph if and only if* n *is prime.*

2. X_n *is strongly regular graph if and only if* n *is a prime power.*

3. X_n *is crown graph (the complete bipartite graph minus 1-factor) if and only if* $n = 2p$ *where* p *is an odd prime.*

3 $\mathcal{A}(X_n) = \mathcal{CC}(X_n)$

In this section we will find the values of n such that X_n is a distance regular graph and we will show that $\mathcal{A}(X_n) = \mathcal{CC}(X_n)$ *for all* n. The following result can be obtained from the main result (Theorem 1.2) of Štefko Miklavič and Primož Potočnik [16] and from the Corollary 2.6.

Theorem 3.1. *Let* X_n *be a unitary Cayley graph. Then* X_n *is distance regular graph if and only if* n *is a prime power or* $n = 2p$*, where* p *is an odd prime.*

Note that in [16], it also is shown that, every distance regular circulant graph is distance transitive. Before proving the next result, recall $A_n = \sum_{k\in U_n} W_n^k$, $W_n^n = I$ and

$$\dim(\mathcal{A}(X_n)) = \begin{cases} \tau(\gamma(n)) = \tau(n), & \text{if } n \text{ is square free} \\ \tau(\gamma(n))+1, & \text{if } n \text{ is not a square free.} \end{cases}$$

Let $\ell = \dim(\mathcal{A}(X_n)) - 1$. Then the set $\{I, A_n, A_n^2, \ldots, A_n^\ell\}$ is a basis for $\mathcal{A}(X_n)$. Now our objective is to find another basis for $\mathcal{A}(X_n)$ with mutually disjoint $0,1$ matrices . For that, for each divisor x of $\gamma(n)$, we define $H_x = \sum_{d|n, \gamma(n/d)=x} A_d$. Then for n is even, it is easy to verify that

$$A_n^{2s} = \sum_{x|\gamma(n),\ x\ is\ even} b_x H_x \tag{1}$$

$$A_n^{2t+1} = \sum_{x|\gamma(n),\ x\ is\ odd} b_x H_x \tag{2}$$

where $1 \leq 2s, 2t+1 \leq \ell,\ b_x \in \mathbb{C}$. If n is odd, then we have

$$A_n^f = \sum_{x|\gamma(n)} b_x H_x \tag{3}$$

where $1 \leq f \leq \ell,\ b_x \in \mathbb{C}$.

Theorem 3.2. *Unitary Cayley graph is a pattern polynomial graph.*

Proof. By definition of adjacency algebra, $\mathcal{A}(X_n)$ is a matrix subalgebra of $M_n(\mathbb{C})$, $I \in \mathcal{A}(X_n)$ and is closed with respect to conjugate transposition. And by definition, X_n is a connected regular graph hence from Lemma 1.1, $\mathbf{J} \in \mathcal{A}(X_n)$. Consequently, it is sufficient to prove, $A(X_n)$ is closed under Hadamard product. But from Theorem 1.3, it is equivalent to showing that $A(X_n)$ has a basis of disjoint $0,1$-matrices.

From the Equations (1), (2) and (3) it follows that for every divisor x of $\gamma(n)$, $H_x \in \mathcal{A}(X_n)$. Hence the set

$$\begin{cases} \{A_d | d \text{ divides } n\}, & \text{when n is square free,} \\ \{I, H_{\gamma(n)} - I\} \cup \{H_x | x \text{ divides } \gamma(n), x \neq \gamma(n)\}, & \text{otherwise} \end{cases}$$

forms *the* basis for $\mathcal{A}(X_n)$ with disjoint $0,1$-matrices. □

Since X_n is a pattern polynomial graph hence it is a distance polynomial graph, walk regular graph, strongly distance-balanced graph, edge regular graph *etc...* for details refer [14].

The following theorem is a consequence of proposition 2.1, in [13], which was first given in [19].

Theorem 3.3. *Let n be a positive integer and $\mathcal{B}_n = \{A_d | d \text{ divides } n\}$. Then $L(\mathcal{B}_n)$ is a coherent subalgebra of $M_n(\mathbb{C})$ of dimension $|\mathcal{B}_n| = \tau(n)$, where $L(S)$ is the linear span of the set S.*

Hence we have $\mathcal{A}(K_n) \subseteq \mathcal{A}(X_n) \subseteq L(\mathcal{B}_n) \subseteq \mathcal{A}(C_n) \subseteq \mathcal{A}(DC_n) \subset M_n(\mathbb{C})$. Also $\mathcal{A}(X_n) = L(\mathcal{B}_n)$ if and only if n is a square free number and $\mathcal{A}(K_n) = \mathcal{A}(X_n) = L(\mathcal{B}_n)$ if and only if n is a prime number.

The following result characterizes integral circulant graphs also given by Wasin So [17].

Corollary 3.4. *A graph X is integral circulant graph if and only if $A(X) \in L(\mathcal{B}_n)$.*

We now associate an integral circulant matrix to an even arithmetical function. Recall an arithmetical function $f(m)$ is said to be even $(mod\ n)$ if $f(m) = f(\gcd(m,n))\ \forall m \in \mathbb{Z}^+$. Let $n(\geq 2)$ be fixed. Let E_n denote the set of all even functions $(mod\ n)$. The following theorem shows that $L(\mathcal{B}_n)$ is isomorphic to E_n as a vector space over $\mathbb{C}$.

Theorem 3.5 (Pentti Haukkanen [9]). *The set E_n forms a complex vector space under usual sum of functions and the scalar multiplication. The dimension of vector space E_n is $\tau(n)$.*

References

[1] Tom M. Apostol, Introduction to Analytic Number theory, *Springer-Verlag*, New York, 1976.

[2] N.L. Biggs, *Algebraic Graph Theory* (second edition), Cambridge University Press, Cambridge (1993).

[3] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance regular Graphs*. Springer-Verlag, (1989).

[4] A. J. Hoffman, *On the polynomial of a graph*, The American Mathematical Monthly, 70 (1): 30-36 (1963).

[5] Dragos M. Cvetkovic, Michael Doob and Horst Sachs, *Spectra of graphs theory and applications*, VEB Deutscher Verlag d. Wiss., Berlin, (1979); Acad. Press, New York, (1979).

[6] Philip J. Davis, *Circulant matrices*, A Wiley-interscience publications,(1979).

[7] G.H Hardy, E.M. Wright, *An introduction to the theory of numbers*, revised by D.R. Heath-Brown, J.H. Silverman, Sixth edition 2008.

[8] A. J. Hoffman, *On the polynomial of a graph*, The American Mathematical Monthly, 70 (1): 30-36 (1963).

[9] Pentti Haukkanen, *An elementary linear algebraic approach to even functions (mod r)*, NAW (Nieuw Archief voor Wiskunde) 5/2 nr.1 maart 29-31(2001).

[10] Walter Klotz and Torsten Sander, *Some properties of unitary Cayley graphs*, The electronic journal of combinatorics, 14 ,R45(2007).

[11] Pieter Moree and Huib Hommerson, *Value distribution of Ramanujan sums and of cyclotomic polynomial coefficients*, arXiv:math/0307352v1 [math.NT] 27 Jul (2003).

[12] Kaoru Motose, *Ramanujan's Sum and cyclotomic polynomials* Math. J. Okayama Univ. 47 , 65-74(2005).

[13] Mikhail E. Muzychuk, *The structure of Rational Schur Rings over Cyclic Groups*, Europ. J. Combinatorics 14, 479-490(1993).

[14] A.Satyanarayana Reddy and Shashank K Mehta, *Pattern polynomial graphs*, http://arxiv.org/abs/1106.4745.

[15] S. S. Shrikhande and Bhagwandas, *Duals of incomplete block designs*, J. Indian Statist. Assoc. 3(1): 30-37 (1965).

[16] Štefko Miklavič and Primož Potočnik, *Distance-regular circulants*, European Journal of Combinatorics 24,777-784(2003).

[17] Wasin So, *On integral circulant graphs*, Discrete Mathematics 306 153-158 (2005).

[18] Laszlo Toth, *Some remarks on Ramanujan sums and cyclotomic polynomials*, Bull. Math. Soc. Sci. Math. Roumanie Tome 53(101) No. 3,277-292(2010).

[19] Helmut Wielandt, *Finite Permutation Groups*, Academic Press, New York, (1964).