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GEODESIC GRAPHOIDAL COVERING NUMBER OF BICYCLIC GRAPHS

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Abstract: A geodesic graphoidal cover of a graph G is a collection ψ of shortest paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is an exactly one path in ψ. The minimum cardinality of a geodesic graphoidal cover of G is called the geodesic graphoidal covering number of G and is denoted by η_g. In this paper we determine η_g for bicyclic graphs.

Key words: Graphoidal covers, acyclic graphoidal cover, Geodesic Graphoidal cover, bicyclic graphs

1 Introduction

A graph is a pair $G=(V,E)$,where *V* is the set of vertices and *E* is the set of edges. Here we consider only nontrivial, finite, connected, undirected graph without loops or multiple edges. The order and size of *G* are denoted by *p* and *q* respectively. For graph theoretic terminology we refer to Harary [4]. The concept of graphoidal cover was introduced by B.D Acharya and E. Sampathkumar [1] and the concept of acyclic graphoidal cover was introduced by Arumugam and Suresh Suseela [4].The reader may refer [5] and [2] for the terms not defined here.

Let $P=(v_1,v_2,v_3,\ldots,v_r)$ be a path or a cycle in a graph $G=(V,E)$. Then vertices $(v_2,v_3,\ldots,v_{r-1})$ are called internal vertices of P and v_1 and v_r are called external vertices of *P*. Two paths *P* and *Q* of a graph G are said to be internally disjoint if no vertex of *G* is an internal vertex of both P and *Q*.

Definition 1.1 [1] — A graphoidal cover of a graph G is called a collection ψ of (not necessarily open) paths in G satisfying the following conditions:

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ .

(iii) Every edge of G is in exactly one path in ψ .

The minimum cardinality of a graphoidal cover of G is called the graphoidal covering number of G and is denoted by $\eta(G)$.

Definition 1.2 [3] — A graphoidal cover ψ of a graph G is called an acyclic graphoidal cover if every member of ψ is an open path. The minimum cardinality of an acyclic graphoidal cover of G is called the acyclic graphoidal covering number of G and is denoted by $\eta_a(G)$ or η_a.

Definition 1.3 [4] — A geodesic graphoidal cover of a graph G is a collection ψ of shortest paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is an exactly one path in ψ. The minimum cardinality of a geodesic graphoidal cover of G is called the geodesic graphoidal covering number of G and is denoted by η_g.

Definition 1.4 [1] **—** Let ψ be a collection of internally disjoint paths in G. A vertex of G is said to be in the interior of ψ if it is an internal vertex of some path in ψ. Any vertex which is not in the interior of ψ is said to be an exterior vertex of ψ.

Theorem 1.5 [7]— For any graphoidal cover ψ of G, let t_ψ denote the number of exterior vertices of ψ. Let $t = \min t_\psi$ where the minimum is taken over all graphoidal covers of G. Then $\eta = q - p + t$

Corollary 1.6[7] —For any graph G, $\eta \geq q - p$. Morever the following are equivalent.

(i) $\eta = q - p$

(ii) There exists a graphoidal cover without exterior vertices.

(iii) There exists a set of internally disjoint and edge disjoint paths without exterior vertices.

In [4] it is given that $\eta \leq \eta_a \leq \eta_g$ and these inequalities can be strict and also for a tree $\eta = \eta_a = \eta_g = n-1$ and Theorem 1.5 and corollary 1.6 are true for geodesic graphoidal covers.

They observe that $\eta_g = q$ if and only if G is Complete. Further for a cycle $C_m, \eta_g = \begin{cases} 2 \text{ if m is even} \\ 3 \text{ if m is odd} \end{cases}$

Theorem 1.7 [4] —Let G be a unicyclic graph with unique cycle C which is even. Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices on C with degree greater than 2. Then

$$\eta_g = \begin{cases} 2 \text{ if } m = 0 \\ n \begin{pmatrix} \text{if m} \geq 2 \text{ and every } (v,w)\text{-section of C in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path} \end{pmatrix} \\ n+1 \text{ otherwise} \end{cases}$$

Theorem 1.8 [4] — Let G be a unicyclic graph with unique cycle C of odd length 2k+1, $k \geq 1$. Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on C with. Then

$$\eta_g = \begin{cases} 3 & \text{if } m=0 \\ n+2 & \text{if } m=1 \\ n & \left(\begin{array}{l}\text{if m} \geq 2 \text{ and every } (v,w)\text{-section of C in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path}\end{array}\right) \\ n+1 & \text{otherwise} \end{cases}$$

Definition 1.9 [8] — A connected (*p*, *p*+1) graph *G* is called a bicyclic graph.

Definition 1.10 [8] — A one – point union of two cycles is a simple graph obtained from two cycles, say C_l and C_m where $l,m \geq 3$, by identifying one and the same vertex from both cycles. Without loss of generality, we may assume the *l*-cycle to be $u_0u_1\ldots u_{l-1}u_0$ and the *m*-cycle to be $u_0u_lu_{l+1}\ldots u_{m+l-2}u_0$. We denote this graph by *U(l;m)*

Definition 1.11 [8] — A long dumbbell graph is a simple graph obtained by joining two cycles C_l and C_m where $l,m \geq 3$,with a path of length i , $i \geq 1$. Without loss of generality, we may assume $C_l = u_0u_1\ldots u_{l-1}u_0$, $P_i = u_{l-1}u_lu_{l+1}\ldots u_{l+i-1}$ and $C_m = u_{l+i-1}u_{l+i}\ldots u_{l+m+i-2}u_{l+i-1}$ .We denote this graph by *D(l,m,i)*

Definition 1.12 [8] — A cycle with a long chord is a simple graph obtained from an *m*-cycle,

$m \geq 4$, by adding a chord of length l where $l \geq 1$. Let the *m*-cycle be $u_0u_1\ldots u_{m-1}u_0$. Without loss of generality, we may assume the chord joins u_0 with u_i, where $2 \leq i \leq m-2$. That is , $u_0u_mu_{m+1}\ldots u_{l+m-2}u_i$ is the chord. We denote this graph by $C_m(i;l)$

In this paper we determine η_g for bicyclic graphs containing a $U(l;m)$, $D(l,m,i)$, $C_m(i;l)$.

2. Main Results

Theorem 2.1

Let G be a bicyclic graph containing a $U(l,m)$ and both the cycles are of even length. Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on $U(l,m)$. Then

$$\eta_g = \begin{cases} 3 & \text{if } m=0 \\ n+2 & \text{if } m=1 \text{ and} \deg u_k \geq 3, u_k = u_i \\ n+1 & \left(\begin{array}{l}\text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path}\end{array}\right) \\ n+3 & \text{otherwise} \end{cases}$$

Proof:

Let $V(U(l,m)) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_l, u_{l+1}, \ldots, u_{l+m-2}\}$

$V(C_l) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_0\}$

$V(C_m)=\{u_0,u_l,u_{l+1},\ldots,u_{l+m-2},u_0\}$ where *l* and *m* are even.

Case 1: m = 0

Then G = $U(l,m)$

The geodesic graphoidal path double covering is as follows

$P_1=\{u_i,u_{i-1},\ldots,u_1,u_0,u_l,u_{l+1},\ldots,u_j\}$ $[i=\frac{l}{2}\ \&\ j=l+\frac{m}{2}-1]$

$P_2=\{u_i,u_{i+1},\ldots,u_0\}$

$P_3=\{u_0,u_{l+m-2},\ldots,u_j\}$

$\Rightarrow \eta_g \le 3$

Since atleast two vertices on $U(l;m)$ are exterior vertices in any minimum geodesic graphoidal cover sothat $t\ge 2$

Hence $\eta_g \ge q-p+2 \Rightarrow \eta_g \ge 3$

Thus $\eta_g = 3$

Case 2: $m=1$

Let u_k be the unique vertex of degree greater than 2 on $U(l,m)$ other than u_0

Without loss of generality assume that u_k lies on C_l

Sub Case 2a

If $k=\frac{l}{2}$

Let $G_1=G-\{u_1,u_2,\ldots,u_{k-1}\}$ is a unicyclic graph with *n* pendant vertices and m = 1.

By Theorem 1.7 $\eta_g(G_1)=n+1$

Let ψ_1 be a minimum geodesic graphoidal cover of G_1

Clearly any path in ψ_1 is a shortest path in G also and hence

$\psi=\psi_1\cup P$ Where P=$\{u_0,u_1,u_2,\ldots,u_k\}$ is a geodesic graphoidal cover of G.

$\Rightarrow \eta_g(G)\le n+2$

Further all the *n* pendant vertices and at least one vertex on $U(l,m)$ is an exterior point of any minimum geodesic graphoidal cover ψ so that $t \geq n+1$

$$\eta_g(G) = q - p + t \geq n+2$$

$$\therefore \eta_g(G) = n+2$$

Sub Case 2b

If $u_k \neq u_i$

Without loss of generality assume that let $k < \frac{l}{2}$

Let $G_1 = G - \{u_{i+1}, u_{i+2}, ..., u_{l-1}\}$ is a unicyclic graph with $n + 1$ pendant vertices and m = 1.

By Theorem 1.7 $\eta_g(G_1) = n+2$

Let ψ_1 be a minimum geodesic graphoidal cover of G_1

Clealy any path in ψ_1 is a shortest path in G also and hence

$\psi = \psi_1 \cup P$ Where $P = \{u_i, u_{i+1}, u_{i+2}, ..., u_{l-1}, u_0\}$ is a geodesic graphoidal cover of G.

$$\Rightarrow \eta_g(G) \leq n+2+1 = n+3$$

Further all the pendant vertices and at least two vertices on $U(l,m)$ is an exterior points of any minimum geodesic graphoidal cover ψ so that $t \geq n+2$ (u_i & u_j are exterior points)

$$\eta_g(G) = q - p + t \geq 1 + n + 2 \geq n+3$$

$$\therefore \eta_g(G) = n+3$$

Case 3: $m \geq 2$ and there is exactly one (*v*,*w*) section of each of the cycles on $U(l,m)$ in which all the vertices except *v* and *w* have degree 2 and this (*v*,*w*) section is not a shortest path.

Let this (*v*,*w*) section be denoted by $(v = u_s, u_{s+1}, \ldots u_t = w)$ where1< s ,t< $\frac{l}{2}$

Let $G_1 = G - \{u_{i+1}, u_{i+2}, ..., u_{l-1}\}$

Then G_1 is a unicyclic graph with n+1 pendant vertices and m=1

By Case 2 $\eta_g(G_1) = n+1+1 = n+2$

Hence $\eta_g(G) = n+3$

Suppose this (*v*,*w*) section be denoted by

$\left(v = u_s, u_{s+1}, \ldots u_t = w\right)$ where $1 < s < \dfrac{l}{2}, 1 < t < \dfrac{m}{2}$ & u_s lies on C_l, u_t lies on C_m

Then $G_1 = G - \left\{u_{i+1}, u_{i+2}, ..., u_{l-1}\right\}$ is a unicyclic graph with n+1 pendant vertices and m = 2

By Theorem 1.7

$$\eta_g\left(G_1\right) = n+1 \Rightarrow \eta_g\left(G\right) = n+2$$

Case 4: $m \geq 2$ and there is exactly one (*v*,*w*) section of each of the cycles on $U\left(l,m\right)$ in which all the vertices except *v* and *w* have degree 2 and this (*v*,*w*) section is a shortest path.

In this case we prove the result by induction on *n*.

When *n* = 2 , G consists of $U\left(l,m\right)$ and two paths.

These two paths should lie in the different cycles such that $P_1 = \left\{u_i, v_t, v_{t-1}, \ldots, v_1\right\}$ & $P_2 = \left\{u_j, w_t, w_{t-1}, \ldots, w_1\right\}$ where u_i on C_l & u_j on C_m.

Now $G_1 = G - \left\{u_{i+1}, u_{i+2}, ..., u_{l-1}\right\}$ is a unicyclic graph with 2 pentant vertices and m = 2

By Theorem 1.7 $\eta_g\left(G_1\right) = 2$

Let ψ_1 be a minimum geodesic graphoidal cover of G_1

Clearly any path in ψ_1 is a shortest path in G also and hence

$\psi = \psi_1 \cup \left\{u_i, u_{i+1}, ..., u_{l-1}, u_0\right\}$ is a minimum geodesic graphoidal cover of G.

$$\left(i.e.\right)\psi = \left\{\left(v,w\right)\text{Section} \cup \left(u_0, u_i\right)\text{Section} \cup \left(u_0, u_j\right)\text{Section}\right\}$$

$$\Rightarrow \eta_g\left(G\right) = 3 = n+1$$

We now assume that the result is true for all bicyclic graph contains a $U\left(l,m\right)$

Satisfying the condition stated in case 4 with n-1 pendant vertices with $m \geq 2$.

Let G be a bicyclic graph contains a $U\left(l,m\right)$ Satisfying the condition stated in case 4 with *n* pendant vertices where $n \geq 3$ with $m \geq 2$.

Let $P_1 = \left\{u_i, v_t, v_{t-1}, \ldots, v_1\right\}$ be a path in G such that $\deg v_1 = 1, \deg v_2 = \deg v_3 = \cdots = \deg v_t = 2, \& \deg u_i \geq 3$ and P is disjoint from $U\left(l,m\right)$ when m=2.

Let $G_1 = G - \{v_1, v_2 \ldots, v_t\}$ is a bicyclic graph contains a $U(l,m)$ Satisfying the condition stated in case 4 with n-1 pendant vertices with $m \geq 2$.

If every (*v*,*w*) section of each of the cycles on $U(l,m)$ in G_1 in which all the vertices except *v* and *w* have degree 2 is a shortest path then by induction hypothesis $\eta_g(G_1) = n-1+1 = n$

Let ψ be a minimum geodesic graphoidal cover of G_1

Then $\psi \cup \{p\}$ minimum geodesic graphoidal cover of G

$$\Rightarrow \eta_g(G) \leq n+1$$

Suppose there is a (*v*,*w*) section of each of the cycles on $U(l,m)$ say (u_1, u_k) section in G_1 in which all the vertices except u_1 & u_k have degree 2 and this (u_1, u_k) section is not a shortest path then by case 3 $\eta_g(G_1) = n+1$

Let $P = (u_0, u_1, u_2, \ldots, u_i)$ where $1 < i < \frac{l}{2}$ is a shortest path.

Let ψ be a minimum geodesic graphoidal cover of G_1 and let P_1 be a path in ψ where u_i is external. Let Q be the path consisting of all edges of P_1 and *P*

Then $(\psi - \{P_1\}) \cup \{Q\}$ is a geodesic graphoidal cover of G.

$$\Rightarrow \eta_g(G) \leq n+1$$

Further all the *n* pendant vertices on $U(l,m)$ are exterior points of any minimum geodesic graphoidal cover ψ so that $t \geq n$

$$\eta_g(G) = q - p + t \geq n+1$$

$$\therefore \eta_g(G) = n+1$$

Theorem 2.2

Let G be a bicyclic graph containing a $U(l,m)$ and any one of the cycles is of odd length.

Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on $U(l,m)$.

$$\text{Then } \eta_g = \begin{cases} 4 & \text{if } m = 0 \\ n+2 & \left(\begin{array}{l} \text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path} \end{array}\right) \\ n+3 & \text{otherwise} \end{cases}$$

Proof:

Let $V\left(U\left(l,m\right)\right)=\left\{u_0,u_1,u_2,\ldots,u_{l-1},u_l,u_{l+1},\ldots,u_{l+m-2}\right\}$

$V\left(C_l\right)=\left\{u_0,u_1,u_2,\ldots,u_{l-1},u_0\right\}$

$V\left(C_m\right)=\left\{u_0,u_l,u_{l+1},\ldots,u_{l+m-2},u_0\right\}$ where *l* is odd and *m* is even.

Case 1: m = 0

Then G = $U\left(l,m\right)$

The geodesic graphoidal path double covering is as follows

$P_1=\left\{u_i,u_{i-1},\ldots,u_1,u_0,u_l,u_{l+1},\ldots,u_k\right\}$

$P_2=\left\{u_i,u_{i+1}\right\}$

$P_3=\left\{u_{i+1},\ldots,u_0\right\}$

$P_4=\left\{u_0,u_{l+m-2},\ldots,u_k\right\}$ where $[i=\frac{l-1}{2}\,\&\,k=l+\frac{m}{2}-1]$

$\therefore \eta_g \le 4$

Since atleast three vertices on $U(l;m)$ are exterior vertices in any minimum geodesic graphoidal cover sothat $t\ge 3$

Hence $\eta_g \ge q-p+3 \Rightarrow \eta_g \ge 4$

Thus $\eta_g = 4$

For the remaining cases the proof is similar to the Theorem 2.1. Choose the deletion vertices from the odd cycle only so that the graph G_1 always will be a unicyclic graph with even cycle.

Theorem 2.3

Let G be a bicyclic graph containing a $U\left(l,m\right)$ and both the cycles is of odd length.

Let n denote the number of pendant vertices of G and let m denote the number of vertices of degree greater than 2 on $U\left(l,m\right)$. Then

$$\eta_g=\begin{cases}5 & \text{if } m=0\\ n+3 & \left(\begin{array}{l}\text{if } m\ge 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices}\\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path}\end{array}\right)\\ n+4 & \text{otherwise}\end{cases}$$

Proof:

Let $V\left(U\left(l,m\right)\right)=\left\{u_0,u_1,u_2,\ldots,u_{l-1},u_l,u_{l+1},\ldots,u_{l+m-2}\right\}$

$V(C_l) = \{u_0, u_1, u_2, \ldots, u_{l-1}, u_0\}$

$V(C_m) = \{u_0, u_l, u_{l+1}, \ldots, u_{l+m-2}, u_0\}$ where *l* and *m* are odd.

Case 1: m = 0

Then G = $U(l,m)$

The geodesic graphoidal path double covering is as follows

$P_1 = \{u_i, u_{i-1}, \ldots, u_1, u_0, u_l, u_{l+1}, \ldots, u_k\}$

$P_2 = \{u_i, u_{i+1}\}$

$P_3 = \{u_{i+1}, \ldots, u_0\}$

$P_4 = \{u_{k+1}, u_k\}$

$P_5 = \{u_0, u_{l+m-2}, \ldots u_{k+1}\}$ where $[i = \frac{l-1}{2} \ \& \ k = l + \frac{(m-1)}{2} - 1]$

$\therefore \eta_g \leq 5$

Since atleast four vertices on $U(l;m)$ are exterior vertices in any minimum geodesic graphoidal cover sothat $t \geq 4$

Hence $\eta_g \geq q - p + 4 \Rightarrow \eta_g \geq 5$

Thus $\eta_g = 5$

The proof for the remaining cases is similar to that of Theorem 2.1.

From the Theorem 2.1 to Theorem 2.3 we have the following

Theorem 2.4

Let G be a bicyclic graph containing a long dumbbell graph *D(l,m,i)* if both cycles are of even length (or any one of the cycle is even). Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on *D(l,m,i)* .Then

$$\eta_g = \begin{cases} 3 & \text{if } m = 0 \\ n+1 & \left(\begin{array}{l} \text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path} \end{array} \right) \\ n+3 & \text{otherwise} \end{cases}$$

Theorem 2.5

Let G be a bicyclic graph containing a long dumbbell graph *D(l,m,i)* if both cycles are of odd length . Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on *D(l,m,i)* .Then

$$\eta_g = \begin{cases} 5 & \text{if } m = 0 \\ n+3 & \left(\begin{array}{l}\text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path}\end{array}\right) \\ n+4 & \text{otherwise} \end{cases}$$

Theorem 2.6

Let G be a bicyclic graph containing a $C_m(i;l)$ if both cycles are of even length . Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on $C_m(i;l)$.Then

$$\eta_g = \begin{cases} 3 & \text{if } m = 0 \\ n+1 & \left(\begin{array}{l}\text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path}\end{array}\right) \\ n+3 & \text{otherwise} \end{cases}$$

Theorem 2.7

Let G be a bicyclic graph containing a $C_m(i;l)$ if both cycles are of odd length . Let *n* denote the number of pendant vertices of G and let *m* denote the number of vertices of degree greater than 2 on $C_m(i;l)$.Then

$$\eta_g = \begin{cases} 4 & \text{if } m = 0 \\ n+1 & \left(\begin{array}{l}\text{if } m \geq 2 \text{ and every } (v,w)\text{-section of } C \text{ in which all vertices} \\ \text{except } v \text{ and } w \text{ have degree 2 is a shortest path}\end{array}\right) \\ n+3 & \text{otherwise} \end{cases}$$

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