Volume 1, No. 1, January 2013

Journal of Global Research in Mathematical Archives

RESEARCH PAPER

Available Online at http://www.jgrma.info

MULTI- DIMENSIONAL LAPLACE TRANSFORM FOR NON - HOMOGENOUS PARTIAL DIFFERENTIAL EQUATIONS.

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Abstract: In this work, we implement multi-dimensional Laplace transform method for solving the non – homogenous second order partial differential equations. The results reveal that the Laplace transforms method is very convenient and effective.

Key words and Phrases: Two-dimensional Laplace transforms; Heat equation; second order linear differential equations; Wave equation.

Mathematics Subject Classification2010: Primary 44A30, Secondary 35L05

INTRODUCTION AND NOTATION

In 1990 [6] R.S.Dahiya and Vinayagomorthy established several new theorems and corollaries for calculating Laplace transforms pairs of n – dimensions. They also considered two boundary value problems. The first was related to heat transfer for cooling off a very thin semi – infinite homogenous plate into the surrounding medium solved by using double Laplace transforms, the second, was heat equation for the semi – infinite slab where the sides of the slab are maintained at prescribed temperature.

In [5](1992) J.Saberi Najafi and R.S.Dahiya established several new theorems for calculating Laplace transforms of n-dimensions and in the second part ,application of those theorems to a number of commonly used special functions was considered, and finally, one-dimensional wave equation involving special functions was solved by using two dimensional Laplace transforms.

Later in (1999) R.S.Dahiya proved certain theorems involving the classical Laplace transforms of N-variables and in the second part a non-homogenous partial differential equations of parabolic type with some special source function was considered.

Recently in [1],[2],[3](2004,2006,2008) authors, established new theorems and corollaries involving systems of two - dimensional Laplace transforms containing several equations.

In [4], certain type of time fractional heat equation is solved via one dimensional Laplace transform.

DEFINITIONS AND NOTATIONS

The generalization of the well-known Laplace transform

$$L\{f(t);s\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

to n-dimensional is given by

$$L_n\{f(\bar{t});\bar{s}\} = F(\bar{s}) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \exp(-\bar{s}.\bar{t}) f(\bar{t}) P_n(d\bar{t}),$$

where $\bar{t} = (t_1, t_2, ..., t_n), \bar{s} = (s_1, s_2, ..., s_n), \bar{s}.\bar{t} = \sum_{i=1}^{n} s_i t_i, P_n(d\bar{t}) = \prod_{k=1}^{n} dt_k$, and f is a function from R_+^n to C with this property: if at least one component t_j of t to be negative then f is equal to zero.

The inverse Laplace transform

$$L^{-1}\{F(s);t\} = f(t) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} e^{st} F(s)\,ds,$$

to n-dimensional is given by

$$L_n^{-1}\{F(\bar{s});\bar{t}\} = f(\bar{t}) = (2\pi i)^{-n} \int_{c_n-i\infty}^{c_n+i\infty} \int_{c_{n-1}-i\infty}^{c_{n-1}+i\infty} \ldots \int_{c_1-i\infty}^{c_1+i\infty} \exp(\bar{s}.\bar{t}) F(\bar{s}) P_n(d\bar{s}),$$

Where $c_k, k = 1, 2, ..., n$ are real number, and $P_n(d\bar{s}) = \prod_{k=1}^{n} ds_k$.

As an application of Laplace transform, we evaluate the integral below by means of Laplace transform.

Lemma 2.1. Calculate the integral

$$f = \int_0^\infty \frac{1}{\sqrt{x}} \prod_{k=1}^{n} J_k(2\sqrt{x})\,dx,$$

where $J_k(.)$ is kth-order Bessel function of first kind.

Solution. Let us introduce the function

$$f(\bar{t}) = f(t_1, t_2, ..., t_n) = \int_0^\infty \frac{1}{\sqrt{x}} \prod_{k=1}^{n} t_k^{\left(\frac{k}{2}\right)} J_k(2\sqrt{t_k x})\,dx, \tag{2.1}$$

by taking the n-dimensional Laplace transform of (2.1), we get

$$F(\bar{s}) = L_n\{f(\bar{t});\bar{s}\} = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \exp(-\bar{s}.\bar{t}) \left\{ \int_0^\infty \frac{1}{\sqrt{x}} \prod_{k=1}^{n} t_k^{\left(\frac{k}{2}\right)} J_k(2\sqrt{t_k x})\,dx \right\} P_n(d\bar{t}),$$

$$= \int_0^\infty \frac{1}{\sqrt{x}} \left\{ \int_0^\infty \int_0^\infty \ldots \int_0^\infty \exp(-\bar{s}.\bar{t}) \prod_{k=1}^{n} t_k^{\left(\frac{k}{2}\right)} J_k(2\sqrt{t_k x}) P_n(d\bar{t}) \right\} dx$$

$$= \int_0^\infty \frac{1}{\sqrt{x}} \left\{ \int_0^\infty \int_0^\infty \ldots \int_0^\infty \prod_{k=1}^{n} \exp(-s_k t_k) t_k^{\left(\frac{k}{2}\right)} J_k(2\sqrt{t_k x}) P_n(d\bar{t}) \right\} dx$$

$$= \int_0^\infty \frac{1}{\sqrt{x}} \prod_{k=1}^{n} \int_0^\infty \exp(-s_k t_k) t_k^{\left(\frac{k}{2}\right)} J_k(2\sqrt{t_k x}) dt_k \, dx ,$$

the inner integral is equal to

$$L\left\{ t_k^{\left(\frac{k}{2}\right)} J_k(2\sqrt{t_k x}); s_k \right\} = \frac{x^{\left(\frac{k}{2}\right)} \exp\left(-\frac{x}{s_k}\right)}{s_k^{k+1}}, k = 1, 2, ..., n \qquad (2.2)$$

substitution of relation (2.2) in the above relation, leads to

$$F(\bar{s}) = \int_0^\infty \frac{1}{\sqrt{x}} \prod_{k=1}^{n} \frac{x^{\left(\frac{k}{2}\right)} \exp\left(-\frac{x}{s_k}\right)}{s_k^{k+1}} dx = \frac{1}{\prod_{k=1}^{n} s_k^{k+1}} \int_0^\infty x^{\left(\sum_{k=2}^{n} \frac{k}{2}\right)} \exp\left(-x\left(\sum_{k=1}^{n} s_k^{-1}\right)\right) dx$$

$$= \frac{1}{\prod_{k=1}^{n} s_k^{k+1}} \int_0^\infty x^{\left(\frac{(n-1)(n+2)}{4}\right)} \exp\left(-x\left(\sum_{k=1}^{n} s_k^{-1}\right)\right) dx$$

The above integral is equal to

$$L\left\{ x^{\left(\frac{(n-1)(n+2)}{4}\right)}; \sum_{k=1}^{n} s_k^{-1} \right\} = \frac{\Gamma\left(\frac{(n-1)(n+2)}{4} + 1\right)}{\left(\sum_{k=1}^{n} s_k^{-1}\right)^{\left(\frac{(n-1)(n+2)}{4} + 1\right)}},$$

therefore, we obtain

$$F(\bar{s}) = \frac{1}{\prod_{k=1}^{n} s_k^{k+1}} \times \frac{\Gamma\left(\frac{(n-1)(n+2)}{4} + 1\right)}{\left(\sum_{k=1}^{n} s_k^{-1}\right)^{\left(\frac{(n-1)(n+2)}{4} + 1\right)}},$$

now, taking the inverse Laplace transform of $F(\bar{s})$, leads to

$$f(\bar{t}) = L_n^{-1} \; F(\bar{s}); \bar{t}$$

$$= (2\pi i)^{-n} \int_{c_n - i\infty}^{c_n + i\infty} \int_{c_{n-1} - i\infty}^{c_{n-1} + i\infty} \cdots \int_{c_1 - i\infty}^{c_1 + i\infty} \exp(\bar{s}.\bar{t}) \{ \frac{1}{\prod_{k=1}^{n} s_k^{k+1}} \times \frac{\Gamma\left(\frac{(n-1)(n+2)}{4} + 1\right)}{\left(\sum_{k=1}^{n} s_k^{-1}\right)^{\left(\frac{(n-1)(n+2)}{4} + 1\right)}} \} P_n(d\bar{s}),$$

now, for the choice of $n = 1$, one gets the following

$$F(s_1)=\frac{1}{s_1^{\ 2}}\int_0^\infty \exp\left(-\frac{x}{s_1}\right)dx=\frac{1}{s_1},$$

so the inverse Laplace transform becomes

$$f\ (t_1)=1,\Rightarrow f\ (t_1=1)=\int_0^\infty \frac{1}{\sqrt{x}}J_1(2\sqrt{x}\,)dx\ =1.$$

The choice of $n=2$ leads to the following

$$F(s_1,s_2)=\frac{1}{s_1^{\ 2}s_2^{\ 3}}\int_0^\infty x\ \exp\left(-x\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\right)dx\ =\frac{1}{s_1^{\ 2}s_2^{\ 3}}\times\frac{\Gamma(2)}{\left(\frac{1}{s_1}+\frac{1}{s_2}\right)^2}=\frac{1}{s_2\ \ (s_1+s_2)^2},$$

therefore, the inverse Laplace transform becomes

$$f\ (t_1,t_2)=\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}\frac{e^{s_2t_2}}{s_2}\left\{\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty}\frac{e^{s_1t_1}}{(s_1+s_2)^2}ds_1\right\}ds_2=\frac{1}{2\pi i}\int_{c_2-i\infty}^{c_2+i\infty}e^{s_2t_2}\frac{t_1e^{-s_2t_1}}{s_2}ds_2$$

$$=t_1H\,(t_2-t_1)=\begin{cases}t_1 & t_2\ge t_1\\ 0 & t_2<t_1\end{cases},$$

Where $L^{-1}\left\{\frac{1}{(s_1+s_2)^2};t_1\right\}=t_1e^{-s_2t_1}$, $L^{-1}\left\{\frac{e^{-s_2t_1}}{s_2};t_2\right\}=H\,(t_2-t_1)$ .we obtain

$$f\ (t_1=1,t_2=1)=\int_0^\infty \frac{1}{\sqrt{x}}J_1(2\sqrt{x}\,)J_2(2\sqrt{x}\,)dx\ =1.$$

SOLUTION TO SECOND-ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The general form of second-order linear partial differential equation in two independent variables is given by

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=r(x\,,y\,),\qquad 0<x\,,y\,<\infty \tag{3.1}$$

Where A,B,C,D,E and F are given constant and $r(x\,,y\,)$ is source function of x and y or constant. We shall use the following for the rest of this section. If

$$\begin{aligned}&u(x\,,0)=f\ (x\,),\qquad u(0,y\,)=g\,(y\,),\qquad u_y\,(x\,,0)=f_1(x\,),\\ &u_x\,(0,y\,)=g_1(y\,),\qquad u(0,0)=u_0,\end{aligned}\tag{3.2}$$

and if their one-dimensional Laplace transformations are $F(p),G(p),F_1(p)\,G_1(q),$

respectively, then

$$L_2\ \ u(x,y);p,q\ \ =\int_0^\infty\int_0^\infty \exp(-px-qy)u(x,y)dx\,dy\ =U(p,q),$$
$$L_2\ \ u_{xx};p,q\ \ =p^2U(p,q)-pG(q)-G_1(q),$$
$$L_2\ \ u_{xy};p,q\ \ =pqU(p,q)-qG(q)-pF(p)+u_0,$$
$$L_2\ \ u_{yy};p,q\ \ =q^2U(p,q)-qF(p)-F_1(p), \tag{3.3}$$
$$L_2\ \ u_x;p,q\ \ =pU(p,q)-G(q),$$
$$L_2\ \ u_y;p,q\ \ =qU(p,q)-F(p),$$
$$L_2\ \ r(x,y);p,q\ \ =R(p,q).$$

Applying double Laplace transformation term wise to partial differential equations and the initial-boundary conditions in (3.2) and using (3.3), we obtain the transformed problem

$$U(p,q)=\frac{1}{Ap^2+Bpq+Cq^2+Dp+Eq+F}\ \ A\ \ pG(q)+G_1(q)$$
$$+B\ \ qG(q)+pF(p)-u_0\ \ +C\ \ qF(p)+F_1(p) \tag{3.4}$$
$$+DG(q)+EF(p)+R(p,q)\ \ .$$

In the following, we state some lemmas with their proof and then illustrate the above method by an example.

Lemma 3.1. Let $f\ (x,y)$ be function for $x\geq 0, y\geq 0$ and $\alpha>0,a,b,\beta$ be constants .Then

$$L_2\ \ \int_0^x\int_0^\xi f\ (\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))\exp(-b(x-\xi))J_0(2\sqrt{\beta\eta(x-\xi)})d\eta d\xi;p,q$$
$$=\frac{F(p,q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta}. \tag{3.5}$$

Proof.

$$\int_0^\infty\int_0^\infty \exp(-px-qy)\{\int_0^x\int_0^\xi f\ (\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))\exp(-b(x-\xi))J_0(2\sqrt{\beta\eta(x-\xi)})d\eta d\xi\}dx\,dy$$

$$=\int_0^\infty \exp(-qy)\ \ \int_0^\infty \exp(-x(p+b))\{\int_0^x \exp(b\xi)\int_0^\xi f\ (\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))J_0(2\sqrt{\beta\eta(x-\xi)})d\eta d\xi\}dx\ \ dy$$

$$=\int_0^\infty \exp(-qy)\ \ \int_0^\infty\int_\xi^\infty \exp(-x(p+b))\exp(b\xi)\{\int_0^\xi f\ (\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))J_0(2\sqrt{\beta\eta(x-\xi)})d\eta\}dx\,d\xi\ \ dy$$

$$=\int_0^\infty \exp(-qy)\ \ \int_0^\infty\int_0^\xi f\ (\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))\{\int_\xi^\infty \exp(-px)\exp(-b(x-\xi))J_0(2\sqrt{\beta\eta(x-\xi)})dx\}d\eta d\xi\ \ dy$$ by

the change of variable $x-\xi=u$ in the inner integral, we get

$$\int_0^\infty \exp(-qy)\ \ \int_0^\infty \exp(-p\xi)\int_0^\xi f\ (\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))\{\int_0^\infty \exp(-u(p+b))J_0(2\sqrt{\beta\eta u})du\}d\eta d\xi\ \ dy,$$

Now, using the relation $L\ \ J_0(2\sqrt{at};s\ \ =\frac{1}{s}\exp(\frac{1}{s})$ (3.6) we obtain

$$\int_0^\infty \exp(-qy)\{\int_0^\infty \exp(-p\xi)\int_0^\xi f(\xi-\eta, y-\alpha\eta)\exp(-\eta(a\alpha+b))\{\frac{\exp\left(-\frac{\beta\eta}{p+b}\right)}{p+b}\}d\eta d\xi\}dy$$

$$=\frac{1}{p+b}\int_0^\infty \exp(-qy)\left\{\int_0^\infty \exp(-\eta(a\alpha+b)\exp\left(-\frac{\beta\eta}{p+b}\right)\{\int_\eta^\infty f(\xi-\eta, y-\alpha\eta))\exp(-p\xi)d\xi\}d\eta\right\}dy,$$

using the change of variable $\xi-\eta=w$ in the inner integral, we get

$$\frac{1}{p+b}\int_0^\infty \exp(-qy)\left\{\int_0^\infty \exp(-\eta(a\alpha+b))\exp\left(-\frac{\beta\eta}{p+b}\right)\{\int_0^\infty f(w, y-\alpha\eta))\exp(-p(\eta+w))dw\}d\eta\right\}dy$$

$$=\frac{1}{p+b}\int_0^\infty \exp(-pw)\left\{\int_0^\infty \exp(-\eta(a\alpha+b)\exp(-p\eta)\exp\left(-\frac{\beta\eta}{p+b}\right)\int_0^\infty f(w, y-\alpha\eta))\exp(-qy)dy\ \ d\eta\right\}dw$$

$$=\frac{1}{p+b}\int_0^\infty \exp(-pw)\left\{\int_0^\infty \exp(-a\alpha\eta)\exp\left(-\eta\left(\frac{\beta}{p+b}+p+b\right)\right)\int_0^\infty f(w, y-\alpha\eta))\exp(-qy)dy\ \ d\eta\right\}dw,$$

now, by change of variable $y-\alpha\eta=z$ in the inner integral, we obtain

$$\frac{1}{p+b}\int_0^\infty \exp(-pw)\left\{\int_0^\infty \exp(-a\alpha\eta)\exp\left(-\eta\left(\frac{\beta}{p+b}+p+b\right)\right)\int_{-\alpha\eta}^\infty f(w,z))\exp(-q(z+\alpha\eta))dz\ \ d\eta\right\}dw,$$ but, $f(w,z)$ had defined for $w\ge 0, z\ge 0$.Then we have $f(w,z)=0$ for $-\alpha\eta<z<0$.

Therefore, we get

$$\frac{1}{p+b}\int_0^\infty \exp(-pw)\left\{\int_0^\infty \exp\left(-\eta\left(\frac{\beta}{p+b}+p+b+a\alpha\right)\right)\int_0^\infty f(w,z))\exp(-q(z+\alpha\eta))dz\ \ d\eta\right\}dw$$

$$=\frac{1}{p+b}\int_0^\infty \exp(-pw)\left\{\int_0^\infty \exp(-qz)f(w,z)\left\{\int_0^\infty \exp\left(-\eta\left(-\frac{\beta\eta}{p+b}+p+b\right)\right)\exp(-a\alpha\eta)\exp(-q\alpha\eta)d\eta\right\}dz\right\}dw$$

$$=\frac{F(p,q)}{p+b}\int_0^\infty \exp\left(-\eta\left(\frac{\beta}{p+b}+p+b+\alpha(q+a)\right)\right)d\eta=\frac{F(p,q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta}.$$

Where $F(p,q)=L_2\ f(x,y);p,q$.

Lemma 3.2. Let $f(y)$ be function for $y\ge 0$ and $\alpha>0, a, b, \beta$ be constant. Then

$$L_2\left\{\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)f(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta;p,q\right\}$$
$$=\frac{F(q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta} \tag{3.7}$$

Proof.

$$\int_0^\infty\int_0^\infty \exp(-px-qy)\left\{\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)f(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta\right\}dx\,dy$$

$$=\frac{1}{\alpha}\int_0^\infty \exp(-qy)\left\{\int_0^\infty \exp(-x(p+b))\left\{\int_0^{\alpha x}\exp(-a\eta)f(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta\right\}dx\right\}dy$$

$$=\frac{1}{\alpha}\int_0^\infty \exp(-qy)\left\{\int_0^\infty \exp(-a\eta)f(y-\eta)\left\{\int_{\frac{\eta}{\alpha}}^\infty \exp(-x(p+b))J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)dx\right\}d\eta\right\}dy,$$

using the change of variable $x-\frac{\eta}{\alpha}=u$ we get

$$\frac{1}{\alpha}\int_0^\infty \exp(-qy)\left\{\int_0^\infty \exp\left(-(p+b)\frac{\eta}{\alpha}\right)\exp(-a\eta)f(y-\eta)\left\{\int_0^\infty \exp\,-(p+b)u\;\;J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}u}\right)du\right\}d\eta\right\}dy,$$

we obtain by virtue of (3.6)

$$\frac{1}{\alpha}\int_0^\infty \exp(-qy)\{\int_0^\infty \exp\left(-(p+b)\frac{\eta}{\alpha}\right)\exp(-a\eta)f(y-\eta)\{\frac{\exp\left(-\frac{\beta\eta}{p+b}\right)}{p+b}\}d\eta\}dy$$

$$=\frac{1}{\alpha(p+b)}\int_0^\infty \exp(-qy)\left\{\int_0^\infty \exp\left(-\eta\left(\frac{p+b}{\alpha}+\frac{\beta}{\alpha(p+b)}\right)\right)\exp(-a\eta)f(y-\eta)d\eta\right\}dy$$

$$=\frac{1}{\alpha(p+b)}\int_0^\infty \exp(-a\eta)\exp\left(-\eta\left(\frac{(p+b)^2+\beta}{\alpha(p+b)}\right)\right)\;\int_0^\infty \exp(-qy)f(y-\eta)dy\;\;d\eta,$$

by the change of variable $y-\eta=w$ we get

$$\frac{1}{\alpha(p+b)}\int_0^\infty \exp(-a\eta)\exp\left(-\eta\left(\frac{(p+b)^2+\beta}{\alpha(p+b)}\right)\right)\;\int_{-\eta}^\infty \exp(-q(\eta+w))f(w)dw\;\;d\eta,$$

but, $f(w)$ defined for $w\geq 0$.Then we have $f(w)=0$ for $-\eta<w<0$.Therefore, we obtain

$$\frac{1}{\alpha(p+b)}\int_0^\infty \exp(-a\eta)\exp\left(-\eta\left(\frac{(p+b)^2+\beta}{\alpha(p+b)}\right)\right)\;\int_0^\infty \exp(-q(\eta+w))f(w)dw\;\;d\eta$$

$$=\frac{1}{\alpha(p+b)}\int_0^\infty \exp(-\eta(q+a))\exp\left(-\eta\left(\frac{(p+b)^2+\beta}{\alpha(p+b)}\right)\right)\;\int_0^\infty \exp(-qw)f(w)dw\;\;d\eta$$

$$= \frac{F(q)}{\alpha(p+b)} \int_0^\infty \exp\left(-\eta\left(\frac{(p+b)^2+\alpha(p+b)(q+a)+\beta}{\alpha(p+b)}\right)\right) d\eta$$

$$= \frac{F(q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta}.$$

Corollary 3.2.1. Setting $F(q)=1$ in the relation (3.7) and subject to $L\left\{\delta(y);q\right\}=1$,

then $f(y)=\delta(y)$ and we get

$$L_2^{-1}\left\{\frac{1}{(p+b)^2+\alpha(p+b)(q+a)+\beta};x,y\right\}$$

$$= \frac{1}{\alpha}\int_0^{\alpha x} \exp(-bx-a\eta)\,\delta(y-\eta)\,J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right) d\eta$$

$$= \frac{1}{\alpha}\int_{y-\alpha x}^{y} \exp(-bx-a(y-z))\,\delta(z)\,J_0\left(2\sqrt{\frac{\beta(y-z)}{\alpha}\left(x-\frac{(y-z)}{\alpha}\right)}\right) dz$$

$$= \begin{cases} 0 & y>\alpha x \\ \dfrac{\exp(-bx-ay)}{\alpha} J_0\left(2\sqrt{\dfrac{\beta y}{\alpha}\left(x-\dfrac{y}{\alpha}\right)}\right) & y<\alpha x \end{cases}$$

$$= \begin{cases} 0 & y>\alpha x \\ \dfrac{\exp(-bx-ay)}{\alpha} J_0\left(\dfrac{2}{\alpha}\sqrt{\beta y(\alpha x-y)}\right) & y<\alpha x \end{cases}. \qquad (3.8)$$

Along with the change of variable $y-\eta=z$.

Corollary 3.2.2. Using the following relations

$$L_2\left\{\frac{\partial g(x,y)}{\partial x};p,q\right\} = pG(p,q) - L\left\{g(0,y);q\right\},$$

$$L_2\left\{\frac{\partial g(x,y)}{\partial y};p,q\right\} = qG(p,q) - L\left\{g(x,0);p\right\},$$

Leads to

$$L_2^{-1}\left\{pG(p,q);x,y\right\} = \frac{\partial g(x,y)}{\partial x} + g(0,y)\,\delta(x),$$

$$L_2^{-1}\left\{qG(p,q);x,y\right\} = \frac{\partial g(x,y)}{\partial y} + g(x,0)\,\delta(y).$$

In the relation (3.7), we set

$$G(p,q)=\frac{F(q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta},$$

therefore we have

$$g(x,y)=\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)f(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta,$$

$$g(0,y)=0,$$

$$g(x,0)=\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)f(-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta=0,$$

subject to $0<\eta<\alpha x$ and $f(y)$ is defined for $y>0$ then $f(-\eta)=0$.

Utilizing the Leibnitz's theorem and considering that

$$\frac{d}{dx}(J_0(u(x))=-u'(x)J_1(u(x)) \qquad \textit{and} \qquad J_0(0)=1,$$

we have

$$\frac{\partial g(x,y)}{\partial x}=\exp(-bx-a\alpha x)f(y-\alpha x)+$$

$$-\frac{b}{\alpha}\int_0^{\alpha x}f(y-\eta)\exp(-(a\eta+bx))J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta+$$

$$-\frac{1}{\alpha}\int_0^{\alpha x}f(y-\eta)\exp(-(a\eta+bx))\sqrt{\frac{\beta\eta}{\alpha\left(x-\frac{\eta}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta\eta}{\alpha}(x-\frac{\eta}{\alpha})}\right)d\eta,$$

$$\frac{\partial g(x,y)}{\partial y}=\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)\frac{\partial f(y-\eta)}{\partial y}d\eta.$$

Therefore

$$L_2^{-1}\left\{\frac{pF(q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta};x,y\right\}=L_2^{-1}\ pG(p,q);x,y$$

$$=\exp(-x(b+a\alpha))f(y-\alpha x)-\frac{b}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)f(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta$$

$$-\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)f(y-\eta)\sqrt{\frac{\beta\eta}{\alpha\left(x-\frac{\eta}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta, \qquad (3.9)$$

and

$$L_2^{-1}\left\{\frac{qF(q)}{(p+b)^2+\alpha(p+b)(q+a)+\beta};x,y\right\}=L_2^{-1}\ \ qG(p,q);x,y$$

$$=\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)\frac{\partial f\ (y-\eta)}{\partial y}d\eta. \qquad (3.10)$$

Lemma 3.3. Let $f\ (x)$ be function for $x\geq 0$and $\alpha>0,a,b,\beta$ be constant and $y<\alpha x$ .Then

$$L_2\left\{\frac{\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f\ (\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi;p,q\right\} \qquad (3.11)$$
$$=\frac{F(p)}{(p+b)^2+\alpha(p+b)(q+a)+\beta}.$$

Proof.

$$\int_0^\infty\exp(-qy)\int_0^\infty\exp(-px)\left\{\frac{\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f\ (\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi\right\}dx\,dy$$

$$=\frac{1}{\alpha}\int_0^\infty\exp(-y\,(q+a))\int_0^\infty\exp(b\xi)f\ (\xi)\left\{\int_{\xi+\frac{y}{\alpha}}^\infty\exp(-x\,(p+b))J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)dx\right\}d\xi\,dy$$, using the variable $x-\xi-\dfrac{y}{\alpha}=u$ we obtain

$$\frac{1}{\alpha}\int_0^\infty\exp(-y\,(q+a))\int_0^\infty\exp(b\xi)f\ (\xi)\left\{\int_0^\infty\exp\left(-\left(u+\xi+\frac{y}{\alpha}\right)(p+b)\right)J_0\left(2\sqrt{\frac{\beta y}{\alpha}u}\right)du\right\}d\xi\,dy$$

$$=\frac{1}{\alpha}\ \int_0^\infty\exp(-p\xi)f\ (\xi)d\xi\ \left\{\int_0^\infty\exp\left(-y\left(q+a+\frac{p+b}{\alpha}\right)\right)\int_0^\infty\exp(-(p+b)u)J_0\left(2\sqrt{\frac{\beta y}{\alpha}u}\right)du\,dy\right\}$$, now, first integral is the Laplace transform of $f\ (\xi)$ that is equal to $F(p)$ .In the second integral, by virtue of (3.6) we get

$$\frac{F(p)}{\alpha}\left\{\int_0^\infty\exp\left(-y\left(q+a+\frac{p+b}{\alpha}\right)\right)\left\{\frac{\exp\left(-\frac{\beta y}{\alpha(p+b)}\right)}{p+b}\right\}dy\right\}$$

$$=\frac{F(p)}{\alpha(p+b)}\left\{\int_0^\infty\exp\left(-y\left(q+a+\frac{p+b}{\alpha}+\frac{\beta}{\alpha(p+b)}\right)\right)dy\right\}$$

$$=\frac{F(p)}{(p+b)^2+\alpha(p+b)(q+a)+\beta}.$$

Corollary 3.3.1. Utilizing the corollary3.2.2 and lemma3.3 and letting

$$G(p,q)=\frac{F(p)}{(p+b)^2+\alpha(p+b)(q+a)+\beta},$$

$$g(x,y)=\frac{\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi,$$

we have

$$g(0,y)=\frac{\exp(-ay)}{\alpha}\int_0^{-\frac{y}{\alpha}}\exp(b\xi)f(\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(-\xi-\frac{y}{\alpha}\right)}\right)d\xi=0,$$

where $f(\xi)=0$ for $-\frac{y}{\alpha}<\xi<0$. Direct application of the Leibnitz's Theorem leads to

$$\frac{\partial g(x,y)}{\partial x}=\frac{-b\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi+$$

$$+\frac{\exp(-bx-ay)}{\alpha}\exp\left(b\left(x-\frac{y}{\alpha}\right)\right)f\left(x-\frac{y}{\alpha}\right)J_0(0)+$$

$$-\frac{\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)\sqrt{\frac{\beta y}{\alpha\left(x-\xi-\frac{y}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi,$$

then

$$L_2^{-1}\left\{\frac{pF(p)}{(p+b)^2+\alpha(p+b)(q+a)+\beta};x,y\right\}=L_2^{-1}\ pG(p,q);x,y$$

$$=\frac{1}{\alpha}\exp\left(-y\left(a+\frac{b}{\alpha}\right)\right)f\left(x-\frac{y}{\alpha}\right)$$

$$-\frac{b\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi$$

$$-\frac{\exp(-bx-ay)}{\alpha}\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)\sqrt{\frac{\beta y}{\alpha\left(x-\xi-\frac{y}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi. \qquad (3.12)$$

Problem 3.1. If $A=1, B=\alpha, C=0, D=2b+a\alpha, E=\alpha b, F=b^2+\alpha ab+\beta,$ where $\alpha>0, a, b, \beta$ are constant, then (3.1) reduces to

$$u_{xx}+\alpha u_{xy}+(2b+a\alpha)u_x+\alpha b u_y+(b^2+\alpha ab+\beta)u=r(x,y), \qquad 0<x,y<\infty, \qquad (3.13)$$

with the following initial conditions

$$u(x,0)=f(x), \qquad u(0,y)=g(y), \qquad u_x(0,y)=g_1(y), \qquad u(0,0)=u_0. \qquad (3.14)$$

Solution.

Applying the double Laplace transform to Eq. (3.13), we obtain

$$U(p,q) = \frac{1}{p^2 + \alpha pq + (2b + a\alpha)p + \alpha bq + b^2 + \alpha ab + \beta} \; R(p,q) + pG(q) + G_1(q) + \alpha qG(q) + \alpha pF(p) - \alpha u_0 + (2b + a\alpha)G(q) + \alpha bF(p)$$

$$= \frac{1}{(p+b)^2 + \alpha(p+b)(q+a) + \beta} \; R(p,q) + pG(q) + G_1(q) + \alpha qG(q) + \alpha pF(p) - \alpha u_0 + (2b + a\alpha)G(q) + \alpha bF(p) \; . \quad (3.15)$$

By taking the inverse double Laplace transform of Eq.(3.15), we obtain the solution of Eq.(3.13) as follows

$$u(x,y) = \int_0^x \int_0^{\xi} r(\xi - \eta, y - \alpha\eta) \exp(-\eta(a\alpha + b)) \exp(-b(x - \xi)) J_0(2\sqrt{\beta\eta(x-\xi)}) d\eta \, d\xi$$

$$+ \exp(-x(b + a\alpha)) g(y - \alpha x)$$

$$- \frac{b}{\alpha} \int_0^{\alpha x} \exp(-bx - a\eta) g(y - \eta) J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x - \frac{\eta}{\alpha}\right)}\right) d\eta$$

$$- \frac{1}{\alpha} \int_0^{\alpha x} \exp(-bx - a\eta) g(y - \eta) \sqrt{\frac{\beta\eta}{\alpha\left(x - \frac{\eta}{\alpha}\right)}} J_1\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x - \frac{\eta}{\alpha}\right)}\right) d\eta$$

$$+ \frac{1}{\alpha} \int_0^{\alpha x} \exp(-bx - a\eta) g_1(y - \eta) J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x - \frac{\eta}{\alpha}\right)}\right) d\eta$$

$$+ \int_0^{\alpha x} \exp(-bx - a\eta) J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x - \frac{\eta}{\alpha}\right)}\right) \frac{\partial g(y - \eta)}{\partial y} d\eta$$

$$+ \exp\left(-y\left(a + \frac{b}{\alpha}\right)\right) f\left(x - \frac{y}{\alpha}\right)$$

$$-b\exp(-bx-ay)\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi$$

$$-\exp(-bx-ay)\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)\sqrt{\frac{\beta y}{\alpha\left(x-\xi-\frac{y}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi$$

$$+\frac{(2b+a\alpha)}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)g(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta$$

$$+b\exp(-bx-a\eta)\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)J_0\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi$$

$$-\begin{cases}0, & y>\alpha x,\\ u_0\exp(-bx-ay)J_0\left(\frac{2}{\alpha}\sqrt{\beta y(\alpha x-y)}\right), & y<\alpha x,\end{cases}$$

therefore

$$u(x,y)=\int_0^x\int_0^\xi r(\xi-\eta,y-\alpha\eta)\exp(-\eta(a\alpha+b))\exp(-b(x-\xi))J_0(2\sqrt{\beta\eta(x-\xi)})d\eta d\xi$$

$$+\exp(-x(b+a\alpha))g(y-\alpha x)$$

$$-\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)g(y-\eta)\sqrt{\frac{\beta\eta}{\alpha\left(x-\frac{\eta}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta$$

$$+\frac{1}{\alpha}\int_0^{\alpha x}\exp(-bx-a\eta)g_1(y-\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)d\eta$$

$$+\int_0^{\alpha x}\exp(-bx-a\eta)J_0\left(2\sqrt{\frac{\beta\eta}{\alpha}\left(x-\frac{\eta}{\alpha}\right)}\right)\left(\frac{\partial g(y-\eta)}{\partial y}+\left(\frac{b+a\alpha}{\alpha}\right)g(y-\eta)\right)d\eta$$

$$+\exp\left(-y\left(a+\frac{b}{\alpha}\right)\right)f\left(x-\frac{y}{\alpha}\right)$$

$$-\exp(-bx-ay)\int_0^{x-\frac{y}{\alpha}}\exp(b\xi)f(\xi)\sqrt{\frac{\beta y}{\alpha\left(x-\xi-\frac{y}{\alpha}\right)}}J_1\left(2\sqrt{\frac{\beta y}{\alpha}\left(x-\xi-\frac{y}{\alpha}\right)}\right)d\xi$$

$$-\begin{cases}0, & y>\alpha x\\ u_0\exp(-bx-ay)J_0\left(\frac{2}{\alpha}\sqrt{\beta y(\alpha x-y)}\right), & y<\alpha x\end{cases}.$$

CONCLUSION

This article deals with the construction of exact solution to the certain boundary value problems. Although the method is well – suited to solve the second order partial differential equations, the method could lead to a promising approach for many applications in applied sciences.

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