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Further properties on strongly generalized star semi- continuous mappings

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Abstract: The aim of this paper is to introduce and study the class of strongly generalized star semi – closed sets which is weaker than semi-closed sets (Crossly and Hildebrand , 1971) and stronger than both strongly generalized semi-closed sets (El-Maghrabi and Nasef ,2008) and semi generalized-closed sets (Bhattacharya and Lahiri,1987). Also, through this paper some concepts such as: strongly generalized star semi – continuous, strongly generalized star semi –closed and strongly generalized star semi –homeomorphism maps are discussed and investigated via a strongly generalized star semi –closed set.

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INTRODUCTION

In 1970, Levine [15] introduced the concept of generalized closed (briefly, g-closed) sets of a topological space. Bhattacharrya and Lahiri [4] defined and studied the notion of sg- closed sets. In 1990, Arya and Nour [2] introduced the concept of gs- closed sets. Veera Kumar [21] defined and studied the notion of g^*-closed sets. The notion of g^*s - closed sets was defined by El-Maghrabi and Nasef [12]. The purpose of the present paper is to define and investigate the concept of strongly generalized star semi-closed sets. Some notions are introduced and investigated via a strongly generalized star semi-closed set such as : strongly generalized star semi- continuity, strongly generalized star semi-irresoluteness, strongly generalized star semi- closed and strongly generalized star semi- homeomorphism maps.

PRELIMINARIES

Throughout this paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let X be a space and A be a subset of X . The closure of A and the interior of A are denoted by cl (A) and int(A) respectively . A subset A of X is said to be regular-open[19] (resp. semi - open[14], pre-open[17],Q-set[13]) if $A \quad int(cl(A))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$, $int(cl(A)) \quad cl(int(A))$). A subset A of X is said to be semi – closed if $X-A$ is semi – open or, equivalently, if $int(cl(A)) \subseteq A$ [8]. The family of all semi – open (resp. semi-closed) sets will be denoted by $SO(X,\tau)$ (resp. $SC(X,\tau)$). The intersection (resp. the union) of all semi- closed (resp. semi-open) sets containing (resp. contained in) A is called the semi – closure (resp. the semi - interior) of A and will be denoted by $s-cl(A)$ (resp. $s-int(A)$).

Definition 2.1. A subset of a space (X,τ) is called:

1- a generalized closed (briefly, g-closed) [15] set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,

2- a semi generalized-closed (briefly, sg-closed) [4] set if $s-cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi- open,

3- a generalized semi-closed (briefly, gs-closed)[2] set if $s-cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open,

4- a strongly generalized semi-closed (briefly, g^*s-closed) [12] set if $s-cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open, 5- a g^*-closed [21] set if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open.

Remark 2.1. The complement of g-closed (resp.sg- closed, gs-closed, g^*s-closed, g^*-closed) is called g-open (resp. sg-open, gs-open, g^*s - open, g^*-open).

Definition 2.2. A mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is called g-continuous[3] (resp. sg- continuous [20], gs- continuous [11], g^*-continuous [21]) if $f^{-1}(V)$ is g-closed (resp. sg-closed, gs-closed, g^* -closed) in (X,τ) for every closed set V of (Y,σ).

Definition 2.3. A mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be:

(i) g-closed [16] (resp. sg-closed [11], gs- closed [11]) if $f(V)$ is g-closed (resp. sg-closed, gs-closed) in (Y,σ) for every closed set V of (X,τ).

(ii) g-open [16] (resp. sg-open [11], gs- open [11]) if $f(V)$ is g- open (resp. sg- open, gs- open) in (Y,σ) for every open set V of (X,τ).

Definition 2.4. A bijective mapping $f:(X,\tau)\rightarrow(Y,\sigma)$ is said to be:

(i) semi homeomorphism(B) [5] if f is semi-continuous and semi-open,

(ii) semi generalized –homeomorphism[10] (briefly, sg- homeomorphism), if f is sg-continuous and sg-open,

(iii) generalized semi-homeomorphism[10] (briefly, gs- homeomorphism), if f is gs-continuous and gs-open.

Lemma 2.1 [7,8,9]. If A and B are two subsets of X , then the following statements are hold:

(i) s-cl (A) (resp. s- int (A)) is semi – closed (resp. semi- open),

(ii) A is semi – closed (resp. semi – open) iff $A \quad s-cl(A)$ (resp. $A \quad s-int(A)$),

(iii) $s-cl(X-A) \quad X-s-int(A)$ and $s-int(X-A) \quad X-s-cl(A)$,

(iv) $A \subseteq s-cl(A)$, $s-int(A) \subseteq A$,

(v) $s-cl(s-cl(A)) \quad s-cl(A)$.

Corollary 2.1 [1]. Let A be a subset of a space (X,τ) . Then $s-cl(A) \quad A \cup int(cl(A))$.

3. More on strongly g^*s-closed sets.

Definition 3.1 A subset A of a space X is called a strongly generalized star semi-closed (briefly, strongly g^*s-closed) set, if $s-cl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs-open in (X,τ) .

A subset B of a space (X,τ) is called a strongly generalized star semi-open (briefly, strongly g^*s-open) set, if $X-B$ is strongly generalized star semi-closed in (X,τ).

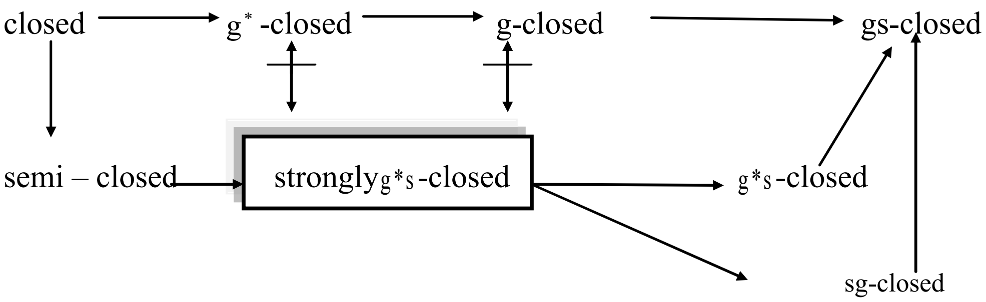
Remark 3.1. The concepts of g-closed (resp. g^*-closed) and strongly g^*s-closed sets are independent.

Example 3.1. If X={a,b,c,d} with two topologies τ_1, τ_2 on X such that : τ_1={X, φ,{a}, {a,b}}, τ_2={X, φ, {a},{b,c},{a,b,c}} , then:

(1) a subset A={b} of X on τ_1 is strongly g^*s- closed but not g-closed and a subset B={a,b,d} of X on τ_1 is g-closed but not strongly g^*s-closed.

(2) a subset C={a} of X on τ_2 is strongly g^*s-closed but not g^*-closed and a subset D={b,d} of X on τ_2 is g^*-closed but not strongly g^*s-closed.

Remark 3.2. By Definition 3.1 and Remark 3.1, we obtain the following diagram.



However , the converses are not true in [2,9,12,21] and by the following examples .

Example 3.2. If $X = \{a,b,c,d\}$ with topologies τ_1 , τ_2 on X such that:
$\tau_1 = \{X, \varphi, \{c,d\}\}$, $\tau_2 = \{X, \varphi, \{c\}, \{c,b\}, \{b,c,d\}\}$, then a subset A= $\{a,b,c\}$ of X on τ_1 is strongly g^*s- closed but not semi–closed. While, a subset B=$\{a,c\}$ of X on τ_2 is g^*s-closed but not strongly g^*s- closed.

Example 3.3. Let X= $\{ a,b,c\}$ with topologies τ_1 , τ_2 on X such that
$\tau_1 = \{ X, \varphi , \{a,b\},\{c\}\}$, $\tau_2 = \{X, \varphi ,\{a\},\{a,b\}\}$. Then, a subset C=$\{a\}$ of X on τ_1 is sg-closed but not stronglyg^*s-closed. But a subset D =$\{a,c\}$ of X on τ_2 is gs-closed but not stronglyg^*s-closed.

Remark 3.3. The union of two stronglyg^*s-closed sets need not be stronglyg^*s-closed. Let X =$\{a,b,c,d\}$ with topology $\tau = \{X, \varphi ,\{a\},\{b\},\{a,b\}\}$. Then, the subsets A=$\{a\}$ and B=$\{b\}$ are stronglyg^*s-closed but their union is not stronglyg^*s-closed.

Theorem 3.1. A subset A of a space (X,τ) is strongly g^*s-closed if and only if every gs - open set G containing A, there exists a semi – closed set F such that $A \subseteq F \subseteq G$.

Proof . Necessity. Let A be a stronglyg^*s-closed set , $A \subseteq G$ and G be gs – open. Then $s-cl(A) \subseteq G$. Set, $s-cl(A) \quad F$. Hence, there exists a semi – closed set F such that $A \subseteq F \subseteq G$.
Sufficiency . Assume that $A \subseteq G$ and G is a gs – open set of X. Then by hypothesis, there exists a semi – closed set F such that $A \subseteq F \subseteq G$, therefore , $s-cl(A) \subseteq G$. So, A is stronglyg^*s-closed.

Theorem 3.2. Let A be a stronglyg^*s-closed set of X . Then $(s-cl(A))-A$ does not contain any non empty g - closed set .
Proof. Let F be a g - closed set such that $F \subseteq (s-cl(A))-A$. Then $F \subseteq X-A$ this implies that $A \subseteq X-F$. Since, A is strongly g^*s- closed and $X-F$ is g - open , then $s\text{-}cl(A) \subseteq X-F$, that is $F \subseteq X-(s-cl(A))$, hence $F \subseteq s-cl(A) \bigcap (X-(s-cl(A))) \quad \varphi$. This shows that $F \quad \varphi$.

The converse of the above theorem may not be true as is shown by the following example.
Example 3.4. In Example 3.1, if A=$\{a,b,d\}$is a subset of X on a topology τ_2 , then $(s-cl(A))-A \quad \{c\}$does not contain any non empty g-closed set.

Corollary 3.1. Let A be a stronglyg^*s-closed set of X . Then $(s-cl(A))-A$ does not contain any non empty gs - closed set .
Proof. Obvious.

Corollary 3.2. Let A be a stronglyg^*s - closed set . Then A is semi - closed if and only if $(s-cl(A))-A$ is gs - closed.
Proof . Necessity . Assume that A is stronglyg^*s - closed and semi - closed sets. Then $s-cl(A) \quad A$and hence $(s-cl(A))-A = \varphi$ which is gs - closed.
Sufficiency. Suppose that $s-cl(A)-A$ is gs - closed and A is stronglyg^*s -closed . Then by Corollary 3.1 , $s-cl(A)-A$ does not contain any non empty gs - closed subset of X. Hence A is semi - closed.

Theorem 3.3. For each $x \in X$, then $\{x\}$ is gs- closed or its complement $X-\{x\}$ is strongly g^*s - closed .

Proof. Suppose that $\{x\}$ is not gs- closed. Then its complement is not gs- open . Since, X is the only gs- open set containing $X-\{x\}$,that is , $s-cl(X-\{x\})\subseteq X$ holds. This implies that $X-\{x\}$ is strongly g^*s - closed .

Proposition 3. 1. If A is a strongly g^*s -closed set and $A\subseteq B\subseteq s-cl(A)$, then B is strongly g^*s - closed.

Proof . Let B $\subseteq$U and U be a gs- open set of X . Then A$\subseteq$U. Since , A is strongly g^*s - closed , hence $s-cl(A)\subseteq U$, but $B\subseteq s-cl(A)$. Then $s-cl(B)\subseteq U$. Hence , B is strongly g^*s - closed .

Proposition 3. 2. If (X,τ) is a topology space and $A\subseteq X$, then A is semi – closed, if one of the following two cases hold :

(1) If A is strongly g^*s-closed and gs-open.

(2) If A is strongly g^*s-closed and open.

Theorem 3.4. Let A be a subset of a space X, the following are equivalent:

(i) A is regular – open,

(ii) A is open and strongly g^*s-closed.

Proof. (i) →(ii). Let U be a gs-open set containing A and A be a regular-open set. Then, $A\bigcup int(cl(A)) \quad A\subseteq U$. So, $s-cl(A)\subseteq U$ and therefore A is strongly g^*s-closed.

(ii)→ (i). Since, A is an open and a strongly g^*s-closed sets, then by Proposition 3.2(2), A is semi-closed. But, A is pre-open . Therefore, A is regular-open.

Theorem 3.5. If A is a subset of a space X, the following are equivalent:

(i) A is clopen,

(ii) A is open, a Q-set and strongly g^*s-closed.

Proof. (i) →(ii). Since, A is clopen, hence A is both open and a Q- set. Let U be a gs-open set containing A. Then, $A\bigcup int(cl(A))\subseteq U$ and so $s-cl(A)\subseteq U$. Hence, A is strongly g^*s-closed.

(ii)→ (i) . Hence by Theorem 3.4, A is regular-open. Since, every regular-open set is open, then A is a Q-set, hence A is closed. Therefore, A is clopen.

Theorem 3.6. For a subset A of a space X, the following statements are equivalent :

(i) A is strongly g^*s - open,

(ii) For each gs-closed set $F\subseteq X$ contained in A, $F\subseteq s-int(A)$,

(iii) For each gs-closed set $F\subseteq X$ contained in A, there exists a semi -open set $G\subseteq X$ such that $F\subseteq G\subseteq A$.

Proof. (i) →(ii). Let $F\subseteq A$ and F be a gs- closed set . Then $X-A\subseteq X-F$ which is gs-open. Hence, $s-cl\ (X-A)\subseteq X-F$. Therefore by Lemma 2.1, (iii) , $F\subseteq s-int(A)$.

(ii) → (iii) . Let $F\subseteq A$ and F be a gs-closed set. Then by hypothesis, $F\subseteq s-int(A)$. Set $s-int(A) \quad G$, hence $F\subseteq G\subseteq A$.

(iii) →(i). Let $X-A\subseteq U$ and U be a gs-open set .Then $X-U\subseteq A$ and by hypothesis, there exists a semi-open set G such that $X-U\subseteq G\subseteq A$, that is , $X-A\subseteq X-G\subseteq U$. Therefore, by Theorem 3.1, $X-A$ is strongly g^*s- closed . Hence, A is strongly g^*s-open.

Lemma 3.1. Let $A\subseteq X$ be a strongly g^*s -closed set. Then $s-cl(A)-A$ is strongly g^*s - open.

Proof. Let F be a gs- closed set such that $F\subseteq(s-cl(A))-A$. Since A is strongly g^*s-closed, then by Corollary 3.1, $F \quad \varphi$. Therefore, $\varphi\subseteq s-int\,(s-cl(A)-A)$. Hence, by Theorem 3.6, $s-cl(A)-A$ is strongly g^*s - open.

4. Strongly g^*s-continuous mappings.

Definition 4.1. A mapping $f:(X,\tau)\to(Y,\sigma)$ is called a strongly generalized star semi-continuous (briefly, strongly g^*s-continuous) mapping if the inverse image of each closed set in Y is strongly g^*s-closed in X.

Definition 4.2. A mapping $f:(X,\tau)\to(Y,\sigma)$ is called strongly generalized star semi-irresolute (briefly, strongly g^*s-irresolute) if, $f^{-1}(U)$ is strongly g^*s-closed in (X,τ), for every strongly g^*s-closed set U of (Y,σ).

Lemma 4.1. (1) Every semi- continuous mapping is strongly g^*s-continuous.

(2) Every strongly g^*s-continuous mapping is sg- continuous (resp. gs-continuous).

Remark 4.1. The concept of strongly g^*s-continuous and g-continuous (resp. g^*-continuous) mappings are independent, as is shown by the following examples.

Example 4.1. Let $X = \{a,b,c,d\}$, $Y = \{a,b,c\}$ with two topologies $\tau_X = \{X, \varphi, \{a\}\}$, $\tau_Y = \{Y, \varphi, \{a\}\}$ and a mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ is defined by $f(a) = b$, $f(b) = c$ and $f(c) = f(d) = a$, then f is g-continuous but not strongly g^*s-continuous.

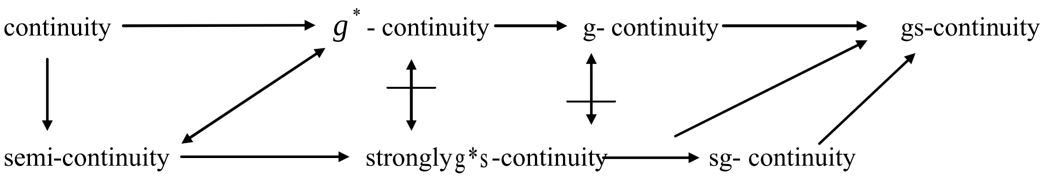
Example 4.2. If $X = Y = \{a,b,c\}$ with topologies.

(i) $\tau_X = \{X, \varphi, \{a\}, \{b\}, \{a,b\}\}$, $\tau_Y = \{Y, \varphi, \{a\}, \{a,b\}\}$,then a mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ which is defined by $f(a) = c$, $f(b) = a$ and $f(c) = b$ is strongly g^*s-continuous but not g-continuous.

(ii) $\tau_X = \{X, \varphi, \{a\}, \{b\}, \{a,b\}\}$, $\tau_Y = \{Y, \varphi, \{b,c\}\}$,then a mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ which is defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ is strongly g^*s-continuous but not g^*-continuous.

(iii) $\tau_X = \{X, \varphi, \{a\}, \{a,b\}\}$, $\tau_Y = \{Y, \varphi, \{b,c\}\}$,then a mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ which is defined by $f(a) = f(c) = a$ and $f(b) = b$ is g^*- continuous but not strongly g^*s-continuous .

Remark 4.2. By Lemma 4.1 and Remark 4.1, we have the following diagram.



The converses of this implication is not true in [6,11,14,21] and by the following examples.

Example 4.3. Let $X = Y = \{a,b,c\}$,with topologies $\tau_X = \{X, \varphi, \{a,b\}\}$, $\tau_Y = \{Y, \varphi, \{b,c\}\}$ and a mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ be defined by $f(a) = f(c) = a$ and $f(b) = b$. Then f is strongly g^*s- continuous but not semi-continuous.

Example 4.4. If $X = \{a,b,c\}$, $\tau_X = \{X, \varphi, \{a,b\}, \{c\}\}$, and a mapping $f:(X, \tau_X) \to (X, \tau_X)$ is defined as $f(a) = a$, $f(b) = c$ and $f(c) = b$, hence f is gs- continuous and sg- continuous but not strongly g^*s- continuous.

Theorem 4.1. A mapping $f:(X,\tau) \to (Y,\sigma)$ is strongly g^*s- continuous iff the inverse image of each open set in Y is strongly g^*s -open in X.

Proof. The necessity. Let $G \subseteq Y$ be an open set. Then, $Y - G$ is closed , hence, by hypothesis, $f^{-1}(Y - G)$ is a strongly g^*s-closed set. Therefore, $f^{-1}(G)$ is strongly g^*s-open.

The sufficiency. Let $F \subseteq Y$ be a closed set . Then, $Y - F$ is open , hence by hypothesis, $f^{-1}(Y - F)$ is a strongly g^*s-open set. Thus $f^{-1}(F)$ is strongly g^*s-closed. So, f is strongly g^*s- continuous.

Lemma 4.2. Every strongly g^*s- irresolute mapping is strongly g^*s-continuous.

Example 4.5. Let $X = Y = \{a,b,c\}$ with two topologies $\tau_X = \{X, \varphi, \{a\}, \{a,b\}\}$, $\tau_Y = \{Y, \varphi, \{a,b\}\}$ and a mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ be defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then, f is strongly g^*s-continuous but not strongly g^*s- irresolute.

Remark 4.3. The composition of two strongly g^*s- continuous mappings may not be strongly g^*s- continuous the following example shows this fact.

Example 4.6. Let $X = Z = \{a,b,c\}$ and $Y = \{a,b,c,d\}$ with the topologies $\tau_X = \{X, \varphi, \{a\}\}$, $\tau_Y = \{Y, \varphi, \{a,c\}\}$, $\tau_Z = \{Z, \varphi, \{c\}\}$, a mapping f from (X, τ_X) to (Y, τ_Y) is the identity map and a mapping $g:(Y, \tau_Y) \to (Z, \tau_Z)$ is defined by $g(a) = a$, $g(b) = g(d) = b$ and $g(c) = c$. Then, f and g are strongly g^*s- continuous, but $g \circ f$ is not strongly g^*s- continuous.

In the next theorem , we give the necessarily condition which satisfying the composition of two strongly g^*s- continuous mappings is also strongly g^*s- continuous.

Theorem 4.2. Let $f:(X, \tau_X) \to (Y, \tau_Y)$ and $g:(Y, \tau_Y) \to (Z, \tau_Z)$ be two mappings . Then, $g \circ f:(X, \tau_X) \to (Z, \tau_Z)$ is strongly g^*s- continuous if one of the following conditions are satisfied.

(i) f is strongly g^*s- continuous and g is continuous,

(ii) f is semi- continuous and g is continuous,

(iii) f is strongly g^*s-irresolute and g is strongly g^*s-continuous.

Proof. (i) Let $F \subseteq Z$ be a closed set and g be a continuous mapping . Then, $g^{-1}(F) \subseteq Y$ is closed . But, f is strongly g^*s- continuous, then $f^{-1}(g^{-1}(F)) \subseteq X$ is strongly g^*s-closed. Therefore, $(g \circ f)^{-1}(F)$ is strongly g^*s-closed in X.

(ii) If V is a closed subset of Z , then $g^{-1}(V) \subseteq Y$ is closed . But, f is semi- continuous, then f is strongly g^*s- continuous, hence $(g \circ f)^{-1}(V)$ is strongly g^*s-closed in X.

(iii) Let V be a closed subset of Z and g is strongly g^*s-continuous. Then, $g^{-1}(V) \subseteq Y$ is strongly g^*s-closed. But, f is strongly g^*s-irresolute, then $f^{-1}(g^{-1}(V)) \subseteq X$ is strongly g^*s-closed. Hence, $g \circ f$ is strongly g^*s-continuous.

5. Strongly g^*s-closed mappings.

Definition 5.1. A mapping $f:(X, \tau) \to (Y, \sigma)$ is called strongly generalized star semi-closed (resp. strongly generalized star semi-open) (briefly, strongly g^*s-closed and strongly g^*s-open) if the image of each closed (resp. open) set of X is strongly g^*s-closed (resp. strongly g^*s-open) in Y.

Remark 5.1. The g-closed (resp. g-open) and strongly g^*s-closed (resp. strongly g^*s-open) mappings are independent. The following examples show this remark.

Example 5.1. Let $X = Y = \{a,b,c,d\}$ and $\tau_X = \{X, \varphi, \{a\}\}$, $\tau_Y = \{Y, \varphi, \{a\}, \{b,c\}, \{a,b,c\}\}$ be two topologies on X,Y respectively . Then, the mapping $f:(X, \tau_X) \to (Y, \tau_Y)$ which is defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ and $f(d) = d$ is g-closed (resp. g-open) but not strongly g^*s-closed (resp. strongly g^*s-open).

Example 5.2. Let $X = Y = \{a,b,c,d\}$ with two topologies $\tau_X = \{X, \varphi, \{b,c,d\}\}$ and $\tau_Y = \{Y, \varphi, \{a\}, \{b,c\}, \{a,b,c\}\}$. Then, the identity mapping from (X, τ_X) into (Y, τ_Y) is strongly g^*s-closed (resp. strongly g^*s-open) but not g-closed (resp. g-open).

Remark 5.2. It is clear that a strongly g^*s-closed (resp. strongly g^*s-open) mapping is weaker than semi-closed (resp. semi-open) and stronger than each of sg-closed (resp.sg-open). The implications between these new types of mappings and other corresponding ones are given by the following diagram.

$$\begin{tikzcd}
\begin{array}{c}\text{closed}\\ \text{(open)}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{g-closed}\\ \text{(g-open)}\end{array} \arrow[r] \arrow[d, leftrightarrow, "|" marking] & \begin{array}{c}\text{gs-closed}\\ \text{(gs-open)}\end{array} \\
\begin{array}{c}\text{semi-closed}\\ \text{(semi-open)}\end{array} \arrow[r] & \boxed{\begin{array}{c}\text{strongly } g^*s\text{-closed}\\ \text{(strongly } g^*s\text{-open)}\end{array}} \arrow[r] & \begin{array}{c}\text{sg-closed}\\ \text{(sg-open)}\end{array} \arrow[u]
\end{tikzcd}$$

The converses of these implications are not true in [11,16,18] and by the following examples.

Example 5.3. If $X \ \ Y \ \ \{a,b,c,d\}$ and $\tau_X \ \ \{X, \varphi, \{a\}, \{a,b\}\}$, $\tau_Y \ \ \{Y, \varphi, \{c,d\}\}$, then a mapping $f:(X,\tau_X)\to(Y,\tau_Y)$ which defined by $f(a) \ \ c$, $f(b) \ \ d$, $f(c) \ \ a$ and $f(d) \ \ b$ is strongly g^*s-closed (resp. strongly g^*s-open) but it is not semi-closed (resp. semi-open).

Example 5.4. If $X \ \ Y \ \ \{a,b,c\}$ with two topologies $\tau_X \ \ \{X, \varphi, \{a,b\}, \{c\}\}$ and $\tau_Y \ \ \{Y, \varphi, \{a\}, \{b,c\}\}$, then a mapping $f:(X,\tau_X)\to(Y,\tau_Y)$ which is defined by $f(a) \ \ a$, $f(b) \ \ c$ and $f(c) \ \ b$ is gs-closed (resp. gs-open) and sg-closed (resp. sg-open) but not strongly g^*s-closed (resp. strongly g^*s-open).

Theorem 5.1. For a bijective mapping $f:(X,\tau)\to(Y,\sigma)$, the following statements are equivalent :

(i) f is strongly g^*s-closed,

(ii) f is strongly g^*s-open,

(iii) f^{-1} is strongly g^*s-continuous.

Proof. (i)→(ii). Let $G \subseteq X$ be an open set . Then, $X-G$ is closed and by hypothesis, $f(X-G)$ is strongly g^*s-closed. Since, f is bijective, hence $Y-f(G)$ is strongly g^*s-closed. Therefore, $f(G)$ is strongly g^*s-open.

(ii)→(iii). If $G \subseteq X$ is an open set, then $f(G)$ is strongly g^*s-open in Y. Since, f is bijective, hence $(f^{-1})^{-1}(G)$ is strongly g^*s-open in Y. Therefore, f^{-1} is strongly g^*s-continuous.

(iii) →(i). Let $F \subseteq X$ be a closed set. Then, $(f^{-1})^{-1}(F)$ is strongly g^*s-closed in Y. But, f is bijective, hence $f(F)$ is strongly g^*s-closed in Y. So, f is strongly g^*s-closed.

Theorem 5.2. A mapping $f:(X,\tau)\to(Y,\sigma)$ is strongly g^*s-open(resp. strongly g^*s-closed) iff for any subset A in (Y,σ) and any closed (resp. open) set F in (X,τ) containing $f^{-1}(A)$, there exists a strongly g^*s-closed (resp. strongly g^*s-open) subset B of (Y,σ) containing A such that $f^{-1}(B) \subseteq F$.

Proof. The necessity. Let $f:(X,\tau)\to(Y,\sigma)$ be a strongly g^*s-open mapping and F be a closed set containing $f^{-1}(A)$ where $A \subseteq Y$. Then, $f(X-F)$ is strongly g^*s-open in Y. Set, $Y-f(X-F) \ \ B$. Since, $f^{-1}(A) \subseteq F$, hence $X-F \subseteq X-f^{-1}(A)$, therefore, $f(X-F) \subseteq Y-A$. Then, $A \subseteq Y-f(X-F) \ \ B$, where, $B \ \ Y-f(X-F)$, then $f^{-1}(B) \ \ f^{-1}(Y-f(X-F)) \ \ X-(f^{-1}f(X-F)) \subseteq F$. Hence, $f^{-1}(B) \subseteq F$.

The sufficiency. Let U be an open set in X. Then, $X-U$ is closed such that $f^{-1}(Y-f(U)) \subseteq X-U$. By hypothesis, there exists a strongly g^*s-closed set B containing $Y-f(U)$, that is, $Y-f(U) \subseteq B$ ….(1). Also, since, $f^{-1}(B) \subseteq X-U$, then $f(U) \subseteq f(X-f^{-1}(B)) \subseteq Y-B$ this implies that $B \subseteq Y-f(U)$…(2) . Hence, from (1),(2) we have $B \ \ Y-f(U)$ which is strongly g^*s-closed. So, $f(U)$ is strongly g^*s-open. Therefore, $f:(X,\tau)\to(Y,\sigma)$ is strongly g^*s-open.

By similarly, we can prove this theorem for a case, if, $f:(X,\tau)\to(Y,\sigma)$ is strongly g^*s-closed.

Remark 5.3. The composition of two strongly g^*s-closed (resp. strongly g^*s-open) mappings may not be strongly g^*s-closed (resp. strongly g^*s-open). The following examples show this fact.

Example 5.5. Let $X \ \ Y \ \ Z \ \ \{a,b,c,d\}$ with topologies $\tau_X \ \ \{X, \varphi, \{a\}, \{a,b\}, \{a,c,d\}\}$, $\tau_Y \ \ \{Y, \varphi, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $\tau_Z \ \ \{Z, \varphi, \{c,d\}\}$. Then, a mapping $f:(X,\tau_X)\to(Y,\tau_Y)$ which defined by $f(a) \ \ a$, $f(b) \ \ d$, $f(c) \ \ b$ and $f(d) \ \ c$ and a mapping $g:(Y,\tau_Y)\to(Z,\tau_Z)$ which also defined by $g(a) \ \ g(b) \ \ a$, $g(c) \ \ c$ and $g(d) \ \ b$ are strongly g^*s-closed, but $g \circ f$ is not strongly g^*s-closed.

Example 5.6. Let $X \ \ Y \ \ Z \ \ \{a,b,c,d\}$ with topologies $\tau_X \ \ \{X, \varphi, \{a\}, \{a,b\}, \{a,c,d\}\}$, $\tau_Y \ \ \{Y, \varphi, \{a\}, \{b,c\}, \{a,b,c\}\}$ and $\tau_Z \ \ \{Z, \varphi, \{c,d\}\}$. Then, a mapping $f:(X,\tau_X)\to(Y,\tau_Y)$ which defined by $f(a) \ \ a$, $f(b) \ \ d$, $f(c) \ \ c$ and

$f(d) = b$ and a mapping $g:(Y,\tau_Y)\to(Z,\tau_Z)$ which also defined by $g(a) = g(c) = c$, $g(b) = d$ and $g(d) = b$ are strongly g^*s -open, but $g\circ f$ is not strongly g^*s-open.

In the following , we give the conditions under which the composition of two strongly g^*s-closed (resp. strongly g^*s-open) may be strongly g^*s-closed (resp. strongly g^*s-open).

Theorem 5.3. Let $f:(X,\tau_X)\to(Y,\tau_Y)$ and $g:(Y,\tau_Y)\to(Z,\tau_Z)$ be two mappings. Then, the following statements are hold:

(i) If, f is closed (resp. open) and g is strongly g^*s-closed (resp. strongly g^*s-open), then $g\circ f$ strongly g^*s-closed (resp. strongly g^*s-open).

(ii) If $g\circ f$ is strongly g^*s-closed (resp. strongly g^*s-open) and f is surjective continuous, then g is strongly g^*s-closed (resp. strongly g^*s-open).

(iii) If $g\circ f$ is closed (resp. open) and g is injective strongly g^*s-continuous then, f is strongly g^*s-closed (resp. strongly g^*s-open).

Proof. (i) Let G be a closed subset of X. Then, $f(G)$ is closed in Y. But , g is strongly g^*s-closed, then $g(f(G))$ is strongly g^*s-closed in Z. Therefore, $g\circ f(G)$ is strongly g^*s-closed.

(ii) If F is closed set in Y, then $f^{-1}(F)$ is closed in X. Hence, by hypothesis, $(g\circ f)(f^{-1}(F))$ is strongly g^*s-closed. Since, f is surjective, then $g(F)$ is strongly g^*s-closed. Therefore, g is strongly g^*s-closed.

(iii) If F is closed set in X, then $g\circ f(F)$ is closed in Z. Hence, by hypothesis, $g^{-1}((g\circ f)(F))$ is strongly g^*s-closed. Since, g is injective, then $f(F)$ is strongly g^*s-closed. Therefore, f is strongly g^*s-closed.

6. strongly g^*s-homeomorphisms.

Definition 6.1. A bijection $f:(X,\tau)\to(Y,\sigma)$ is called a strongly g^*s-homeomorphism if f is both strongly g^*s-continuous and strongly g^*s-open.

Remark 6.1. (1)Every semi-homeomorphism(B) is strongly g^*s-homeomorphism.

(2) Every strongly g^*s-homeomorphism is sg- homeomorphism (resp. gs-homeomorphism).

The converse of above remark is not true as is shown by the following examples.

Example 6.1. Let $X = Y = \{a,b,c,d\}$ with two topologies $\tau_X = \{X,\varphi,\{c,d\}\}$ and $\tau_Y = \{Y,\varphi,\{a\},\{b\},\{a,b\}\}$. Then, a mapping $f:(X,\tau_X)\to(Y,\tau_Y)$ which defined by $f(a) = d$, $f(b) = c$, $f(c) = a$ and $f(d) = b$ is strongly g^*s-homeomorphism but not semi-homeomorphism(B).

Example 6.2. If $X = \{a,b,c\}$ with topology $\tau_X = \{X,\varphi,\{a,b\},\{c\}\}$ and, then a mapping $f:(X,\tau_X)\to(X,\tau_X)$ which defined by $f(a) = a$, $f(b) = c$ and $f(c) = b$ is sg- homeomorphism and gs- homeomorphism but not strongly g^*s- homeomorphism.

By Remark 6.1 and the above examples we obtain the following diagram.

$$\begin{tikzcd} \text{homeomorphism} \arrow[r] & \text{semi- homeomorphism(B)} \arrow[d] & \\ & \boxed{\text{strongly } g^*s\text{- homeomorphism}} \arrow[dl] \arrow[dr] & \\ \text{sg- homeomorphism} \arrow[rr] & & \text{gs- homeomorphism} \end{tikzcd}$$

Proposition 6.1. Let $f:(X,\tau)\to(Y,\sigma)$ be a bijective and strongly g^*s-continuous map. Then, the following statements are equivalent:

(i) f is strongly g^*s -open,

(ii) f is strongly g^*s-homeomorphism,

(iii) f is strongly g^*s-closed.

Proof. (i)→(ii). It is clear from Definition 6.1.

(ii)→(iii). Since, f is strongly g^*s-homeomorphism, then f is strongly g^*s-open. But, f is bijective , hence by Theorem 5.1, f is strongly -g^*s closed.

(iii)→(i). Obvious.

Remark 6.2. The composition of two strongly g^*s- homeomorphism mappings may not be strongly g^*s- homeomorphism. The following example shows this fact.

Example 6.3. Let $X \quad Y \quad Z \quad \{a,b,c\}$ with topologies $\tau_X \quad \{X,\varphi,\{a\}\}$, $\tau_Y \quad \{Y,\varphi,\{a,c\}\}$ and $\tau_Z \quad \{Z,\varphi,\{c\}\}$ .Then, a mapping $f:(X,\tau_X)\to(Y,\tau_Y)$ which defined by $f(a) \quad c$, $f(b) \quad b$ and $f(c) \quad a$ and a mapping $g:(Y,\tau_Y)\to(Z,\tau_Z)$ which also defined by $g(a) \quad c$, $g(b) \quad b$ and $g(c) \quad a$ are strongly g^*s-homeomorphism, but $g\circ f$ is not strongly g^*s-homeomorphism .

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