Volume 1, No. 2, February 2013

Journal of Global Research in Mathematical Archives

ISSN 2320 – 5822

RESEARCH PAPER

Available Online at http://www.jgrma.info

COUPLED COINSIDENCE POINT THEOREM FOR NONLINEAR CONTRACTION IN PARTIALLY ORDERED METRIC SPACES

*K.L. Bondar[1], A.B. Jadhao[2], S.T. Patil[3]

[1]NES Science College, Nanded-431602(MS) INDIA

Email: klbondar_75@rediffmail.com[1]

[2]DSM College, Parbhani(MS)INDIA

Email: abjadhav@rediffmail.com[2]

[3]D.Y. Patil College of Engineering, Akurdi, Pune INDIA

Email: stpatil_49@yahoomail.com[3]

Abstract: The existence theorem of coupled coincidence point is proved. The main tool in the proof of result combines the ideas in the contraction principle with those in the monotonic iterative technique. An example is given satisfying contractive type condition.

INTRODUCTION

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions [1, 6, 7, 13]. Recently Agarwal [1], Bhaskar and Laxmikantham [4], Hussain [11], Nietz and Rodriguez-Lobez [13] presented some new results in partially ordered metric spaces. D. Guo and Laxmikantham [9, 10] introduced the concept of mixed monotone operator and the coupled fixed points. Later on several authors [5, 8, 11, 14, 15] have used this concept and proved the existence of coupled fixed points for mixed monotone operators. Ciric and Laxmikantham [8] generalized the concept of mixed monotone to mixed g-monotone and have obtained existence theorems for coupled fixed points.

In this paper, the existence theorem of coupled coincidence point is proved. The main tool in the proof of result combines the ideas in the contraction principle with those in the monotonic iterative technique. An example is given satisfying contractive type condition.

Recall that if $(X, \leq)$ is a partially ordered set and $f : X \to X$ such that for $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$, then a mapping f is said to be nondecreasing. Similarly, a nonincreasing mapping is defined. Bhaskar, Laxmikantham [2] introduced the following notion of mixed monotone mapping and a coupled fixed point.

Definition 1.1[4] Let $(X, \leq)$ be a partially ordered set and $f : X{\times}X \to X$. The mapping f is said to have mixed monotone property if f is nondecreasing monotone in its first argument and is nonincreasing in its second argument, i.e. for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow f(x_1, y) \leq f(x_2, y);$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow f(x, y_1) \leq f(x, y_2).$$

Definition 1.2[4] An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $f : X \times X \to X$ if $x = f(x, y), y = f(y, x)$.

If there is $x \in X$ such that $x = f(x, x)$ then x is called a fixed point of f. Let S denote the class of functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

In this paper the existence of coupled coincidence points are obtained with the help of the altering functions. We recall the definition of altering function.

Definition 1.3[6] An altering function is a function ψ: $[0, \infty) \to [0, \infty)$ which satisfies following.

(a) ψ is continuous and nondecreasing.

(b) $\psi(t) = 0$ if and only if $t = 0$.

Definition 1.4 A function $\psi : [0, \infty) \to [0, \infty)$ is said to be convex if

$$\psi(\alpha t_1 + (1 - \alpha)t_2) \leq \alpha\, \psi(t_1) + (1 - \alpha)\, \psi(t_2)$$

where $t_1, t_2 \in [0, \infty)$ and $\alpha \in [0, 1]$.

Definition 1.5 A function $\psi: [0, \infty) \to [0, \infty)$ is said to be affine if

$$\psi(\alpha t_1 + (1 - \alpha)t_2) = \alpha\, \psi(t_1) + (1 - \alpha)\, \psi(t_2)$$

where $t_1, t_2 \in [0, \infty)$ and $\alpha \in [0, 1]$.

Altering functions have been used in metric fixed point theory in recent papers [6, 2, 12, 3].

2 Main Results

Analogous with definition 1.1, Laxmikantham and Ciric [8] introduced following concept of mixed g-monotone mapping.

Definition 2.1[8] Let $(X, \leq)$ be a partially ordered set and $f : X{\times}X \to X$ and $g : X{\times}X \to X$. We say that mapping f has mixed g-monotone property if f is nondecreasing g-monotone in its first argument and is nonincreasing g-monotone in its second argument, i.e. for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow f(x_1, y) \leq f(x_2, y);$$

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow f(x, y_1) \leq f(x, y_2).$$

Note that if g is identity mapping, then definition 2.1 reduces to definition 1.1.

Definition 2.2[8] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $f : X{\times}X \to X$ and $g : X{\times}X \to X$ if $g(x) = f(x, y)$; $g(y) = f(y, x)$.

Definition 2.3[8] Let X be a nonempty set and $f : X{\times}X \to X$ and $g : X{\times}X \to X$. f and g are said to have commutative property if $g(f(x, y)) = f(gx, gy)$, for all $x, y \in X$.

Suppose (X, d) be a metric space and $f : X{\times}X \to X$, $g : X{\times}X \to X$ such that $f(X \times X) \subset g(X)$. For any $x_0, y_0 \in X$, we can choose x_1, $y_1 \in X$ such that $g(x_1) = f(x_0, y_0)$ and $g(y_1) = f(y_0, x_0)$. Similarly for $x_1, y_1 \in X$, there exists $x_2, y_2 \in X$ such that $g(x_2) = f(x_1, y_1)$ and $g(y_2) = f(y_1, x_1)$. Continuing this process we can construct sequences $\{gx_n\}$ and $\{gx_n\}$in X such that

$$gx_{n+1} = f(x_n, y_n) \text{ and } gy_{n+1} = f(y_n, x_n). \tag{2.1}$$

For the main result we need following assumptions.

(H_1) $f(X \times X) \subset g(X)$.

(H_2) f has mixed g-monotone property.

(H_3) For any $x, y, u, v \in X$,

$$\psi(d(f(x, y), f(u, v))) \leq (d(gx, gu) + d(gy, gv))\, \psi([d(gx, gu) + d(gy, gv)]/2)$$

where $\beta \in S$ and is convex altering function.

Theorem 2.1 Let (X, d) be a metric space and $f : X{\times}X \to X$, $g : X{\times}X \to X$. Suppose (H_3) holds. If $(x, y) \in X \times X$ is a coupled coinsidence of g and f then $gx = gy$. Moreover, if (x, y) and (x_0, y_0) are coupled coinsidences of g and f then $gx = gx_0 = gy = gy_0$.

Proof: Suppose $(x, y) \in X \times X$ is a coupled coinsidence of g and f. Therefore,

$$g(x) = f(x, y);\ g(y) = f(y, x).$$

From (H_3), we get

$$\begin{aligned}\psi(d((gx, gy))) &= \psi(d(f(x, y), f(y, x))) \\ &\leq \beta(d(gx, gy) + d(gy, gx))\psi\left(\frac{d(gx, gy) + d(gy, gx)}{2}\right) \\ &= \beta(2d(gx, gy))\psi(dgx, gy)).\end{aligned}$$

Since ψ is an altering function and $\beta \in S$, from the above inequality we get $d(gx, gy) = 0$ which implies $gx = gy$.

Now suppose (x, y) and $(x`, y`)$ are coupled coincidences of g and f. Therefore,

$$g(x) = f(x, y);\ g(y) = f(y, x).$$

and

$$g(x`) = f(x`, y`);\ g(y`) = f(y`, x`).$$

Moreover we have $gx = gy$ and $gx` = gy`$. From (H_3) we get,

$$\psi(d(gx, gx')) = \psi(d(f(x, y), f(x', y')))$$

$$\leq \beta(d(gx, gx`) + d(gy, gy`))\psi\left(\frac{d(gx, gx`) + d(gy, gy`)}{2}\right)$$

$$= \beta(d(gx, gx`) + d(gx, gx`))\psi\left(\frac{d(gx, gx`) + d(gx, gx`)}{2}\right)$$

$$= \beta(2d(gx, gx'))\psi(d(gx, gx')).$$

Since ψ is an altering function and $\beta \in S$, from the above inequality we get $d(gx, gx`) = 0$ which implies $gx = gx`$.

This completes the proof.

For the main result we need the following lemma.

Lema 2.1 Let $(X, \leq, d)$ be a partially ordered metric space and $f : X \times X \to X$, $g : X \to X$. Assume (H_1) - (H_3) hold. Suppose there exists $x_0, y_0 \in X$ such that

$$gx_0 \leq f(x_0, y_0) \text{ and } f(y_0, x_0) \leq gy_0 \tag{2.2}$$

If the sequences $\{gx_n\}$, $\{gy_n\}$ are defined by (2.1) then

(a) $\{gx_n\}$ is nondecreasing and $\{gy_n\}$ is nonincreasing sequence.

(b)
$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0 \tag{2.3}$$

$$\lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0 \tag{2.4}$$

(c) $\{gx_n\}$ and $\{gy_n\}$ are both Cauchy sequences.

Proof: Hypothesis (H_1) implies that the sequences $\{gx_n\}$ and $\{gy_n\}$ defined by (2.1) exist.

(a) To prove

$$gx_{n+1} \geq gx_n \tag{2.5}$$

and

$$gy_{n+1} \leq gy_n, \tag{2.6}$$

we use mathematical induction. By (2.2), it is obvious that $gx_0 \leq gx_1$ and $gy_0 \geq gy_1$. Thus (2.5) and (2.6) hold for $n = 0$. Suppose now that (2.5) and (2.6) hold for some fixed $n \geq 0$. Then, since $gx_n \leq gx_{n+1}$ and $gy_n \geq gy_{n+1}$ and as f has g-mixed monotone property,

$$gx_{n+1} = f(x_n, y_n) \leq f(x_{n+1}, y_n) \leq f(x_{n+1}, y_{n+1}) = gx_{n+2},$$

$$gy_{n+1} = f(y_n, x_n) \geq f(y_{n+1}, x_n) \geq f(y_{n+1}, x_{n+1}) = gy_{n+2}.$$

Thus by the mathematical induction, we conclude that (2.5) and (2.6) hold for

all $n \geq 0$. Therefore,

$$gx_0 \leq gx_1 \leq gx_2 \leq \ldots. \leq gx_n \leq gx_{n+1} \leq \ldots. \tag{2.7}$$

$$gy_0 \geq gy_1 \geq gy_2 \geq \ldots. \geq gy_n \geq gy_{n+1} \geq \ldots. \tag{2.8}$$

To prove (b), denote $\delta_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})$. Using (H_3) we obtain

$$\psi(d(gx_n, gx_{n+1})) = \psi(d(f(x_{n-1}, y_{n-1}), f(x_n, y_n)))$$

$$\leq \quad \beta(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n))\psi\left(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n}{2}\right)$$

$$\leq \quad \beta(\delta_{n-1})\psi\left(\frac{(\delta_{n-1})}{2}\right)$$

Similarly we obtain,

$$\psi(d(gy_n, gy_{n+1})) \quad \leq \quad \beta(\delta_{n-1})\psi\left(\frac{(\delta_{n-1})}{2}\right)$$

Since ψ is a convex function,

$$\psi\left(\frac{\delta_n}{2}\right) \quad = \quad \psi\left(\frac{d(g(x_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}\right)$$

$$\leq \quad \frac{1}{2}\psi(d(gx_n, gy_n)) + \frac{1}{2}\psi(d(gy_n, gy_{n+1}))$$

$$\leq \frac{1}{2}\beta(\delta_{n-1})\psi\left(\frac{(\delta_{n-1})}{2}\right) + \frac{1}{2}\beta(\delta_{n-1})\psi\left(\frac{(\delta_{n-1})}{2}\right)$$

$$= \quad \beta(\delta_{n-1})\psi\left(\frac{\delta_{n-1}}{2}\right). \tag{2.9}$$

Moreover, is nondecreasing and $\beta \in S$, hence we have $\delta_n / 2 \leq \delta_{n-1} / 2$.

Therefore,

$$\delta_n \leq \delta_{n-1} \tag{2.10}$$

If there exists n_0 such that $\delta_{n0} = 0$ then obviously (2.3), (2.4) hold.

In other case, suppose $\delta_n \neq 0$ for all $\beta \in N$. Then taking into account (2.10), the sequence $\{\delta_n\}$ is decreasing and bounded below. So

$$\lim_{n\to\infty} \delta_n = r \geq 0 \tag{2.11}$$

Assume $r > 0$, then from (2.9), we have

$$\frac{\psi\left(\frac{\delta_n}{2}\right)}{\psi\left(\frac{\delta_{n-1}}{2}\right)} \leq \beta(\delta_{n-1}) < 1. \tag{2.12}$$

Letting $n \to 1$ in the last inequality and by the fact that is continuous, we get $1 \leq \beta(\delta_{n-1}) < 1$.

Therefore

$$\lim_{n\to\infty} \beta(\delta_n) = 1$$

Since $\beta \in S$, $\lim_{n\to 1} \beta(\delta_n) = 0$ and this contradicts to our assumption that $r > 0$. Therefore $r = 0$ and hence (2.3) and (2.4) hold.

To prove (c), it is sufficient to prove following two statements.

(i) At least one of sequences $\{gx_n\}$ and $\{gy_n\}$ is a Cauchy sequence.

(ii) If one of sequences $\{gx_n\}$ and $\{gy_n\}$ is a Cauchy sequence then so is other.

If possible suppose that $\{gx_n\}$ and $\{gy_n\}$ both are not Cauchy sequences. Therefore there exists $\in_1 > 0$ and $\in_2 > 0$ for which we can find subsequences $\{gx_{n(k1)}\}$ and $\{gy_{n(k2)}\}$ such that

$$n(k_1) > m(k_1) > k_1;\ d(gx_{n(k_1)}, gx_{m(k_1)}) \geq \in_1$$

and

$$n(k_2) > m(k_2) > k_2;\ d(gx_{n(k_2)}, gx_{m(k_2)}) \geq \in_2.$$

Suppose $\in = \min\{\in_1, \in_2\}$; $\quad k = \max\{k_1, k_2\}$, then for $n(k) > m(k) \geq k$,

$$d(gx_{n(k)}, gx_{m(k)}) \geq \in \text{ and } d(gy_{n(k)}, gy_{m(k)}) \geq \in.$$

Corresponding to $m(k)$ we can choose smallest $n_1(k)$ and $n_2(k)$ such that

$n_1(k) > m(k)$, $n_2(k) > m(k)$ satisfying $d(gx_{n1(k)}, gx_{m(k)}) \geq \in$; $d(gy_{n2(k)}, gy_{m(k)}) \geq \in$

and

$$d(gx_{n1(k)-1}, gx_{m(k)}) < \in \tag{2.13}$$

$$d(gy_{n2(k)-1}, gy_{m(k)}) < \in. \tag{2.14}$$

Using (2.13) and the triangular inequality, we have

$$\begin{aligned} \in &\leq d(gx_{n1(k)}, gx_{m(k)}) \\ &\leq d(gx_{n1(k)}, gx_{n1(k)-1}) + d(gx_{n1(k)-1}, gx_{m(k)}) \\ &< d(gx_{n1(k)}, gx_{n1(k)-1}) + \in. \end{aligned}$$

Letting $n \to 1$ and using (2.3) we get $\in \leq \lim\limits_{k\to\infty} d(gx_{n1(k)}, gx_{m(k)}) \leq \in$.

Therefore

$$\lim_{k\to\infty} d(gx_{n1(k)}, gx_{m(k)}) = \in. \tag{2.15}$$

Similarly we obtain

$$\lim_{k\to\infty} d(gx_{n2(k)}, gx_{m(k)}) = \in. \tag{2.16}$$

Again, the triangular inequality gives

$$d(gx_{n1(k)}, gx_{m(k)}) \leq d(gx_{n1(k)}, gx_{n1(k)-1}) + d(gx_{n1(k)-1}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{m(k)}).$$

and

$$d(gx_{n1(k)-1}, gx_{m(k)-1}) \leq d(gx_{n1(k)-1}, gx_{n1(k)}) + d(gx_{n1(k)}, gx_{m(k)}) + d(gx_{n(k)}, gx_{m(k)-1}).$$

Letting $k \to \infty$ in above inequality and using (2.3) and (2.15), we have

$$\in \leq 0 + \lim_{k\to\infty} d(gx_{n(k)-1}, gx_{m(k)-1}) + 0$$

And

$$\lim_{k\to k} d(gx_{n1(k)-1}, gx_{m(k)-1} \leq 0 + \in + 0.$$

Therefore,

$$\lim_{k\to k} d(gx_{n1(k)-1}, gx_{m(k)-1}) = \in \tag{2.17}$$

Similarly we obtain

$$\lim_{k\to k} d(gy_{n2(k)-1}, gx_{m(k)-1}) = \in \tag{2.18}$$

Suppose $n_2(k) > n_1(k)$, since is nondecreasing and using (H_3), (2.7) and (2.8) we have

$$\begin{aligned} \psi(d(gx_{n_1(k)}, gx_{m(k)})) &= d(f(x_{n_{11}(k)-1}, y_{n_1(k)-1}),\ f(x_{m_{11}(k)-1}, y_{m_1(k)-1})) \\ &\leq \beta(d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1})) \end{aligned}$$

$$\psi\left(\frac{d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1})}{2}\right)$$

$$\leq \beta(d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1}))$$

$$\psi\left(\frac{d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_2(k)-1}, gy_{m(k)-1})}{2}\right) \tag{2.19}$$

Taking into account (2.15), (2.17), (2.18) and the fact that ψ is continuous, letting $k \to \infty$ in (2.19), we get

$\psi(\in) \quad \leq \quad \beta(d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1})) \leq \psi(\in)$.

As ψ is altering function, $\psi(\in) > 0$, the last inequality gives us

$$\lim_{k\to\infty} \beta(d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1})) = 1.$$

Since $\beta \in S$, this means that

$$\lim_{k\to\infty} d(gx_{n_1(k)-1}, gx_{m(k)-1}) = 0 \tag{2.20}$$

and

$$\lim_{k\to\infty} d(gy_{n_1(k)-1}, gy_{m(k)-1}) = 0$$

From (2.17) and (2.20) we get $\in = 0$ which is a contradiction.

If $n_1(k) > n_2(k)$ then considering $(d(gy_{n2}(k); gy_{m(k)}))$ and adopting same procedure as above, we get

$$\lim_{k\to\infty} d(gx_{n_2k)-1}, gx_{m(k)-1}) = 0$$

and

$$\lim_{k\to\infty} d(gy_{n_2k)-1}, gy_{m(k)-1}) = 0 \tag{2.21}$$

From (2.18) and (2.21) we get $\in = 0$ which is again a contradiction. Therefore, at least one of the sequences $\{g_{xn}\}$ and $\{g_{yn}\}$ must be a Cauchy sequence. This proves statement (i).

To prove statement (ii), assume $\{gy_n\}$ is Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then there exist $\in > 0$ for which we can find subsequence $\{gx_{n(k)}\}$ such that

$$d(gx_{n(k)}, gx_{m(k)}) \geq \in \tag{2.22}$$

for $n(k) > m(k) > k$. Corresponding to $m(k)$ choose $n(k)$ in such a way that it is smallest integer with $n(k) > m(k)$ and satisfying (2.22). Then

$$d(gx_{n(k)-1}, gx_{m(k)}) < \in . \qquad (2:23)$$

By similar procedure as adopted earlier, we obtain

$$\lim_{k\to\infty} d(gx_{n(k)}, gx_{m(k)}) = \in \tag{2.24}$$

and

$$\lim_{k\to\infty} d(gx_{n(k)-1}, gx_{m(k)-1}) = \in \tag{2.25}$$

Now using (H_3) we have

$\psi(d(gx_{n_1(k)}, gx_{m(k)})) = \psi(d(f(x_{n_{11}(k)-1}, y_{n_1(k)-1}), f(x_{m_{11}(k)-1}, y_{m_1(k)-1})))$

$$\leq \beta(d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1}))\ \psi\left(\frac{d(gx_{n_1(k)-1}, gx_{m(k)-1}) + d(gy_{n_1(k)-1}, gy_{m(k)-1})}{2}\right).$$

(2.26)

Since $\{gy_n\}$ is a Cauchy sequence, we have

$$\lim_{k\to\infty} d(gy_{n(k)-1}, gy_{m(k)-1}) = 0. \tag{2.27}$$

Now ψ is continuous and nondecreasing and using (2.24), (2.25), (2.27); taking limit $k \to \infty$ in (2.26) we obtain

$$\psi(\in) \leq \lim_{k\to\infty} \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1}))\psi\left(\frac{\in +0}{2}\right)$$

$$\leq \lim_{k\to\infty} \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1}))\psi(\in).$$

Since ψ is altering function, $\psi(\in) > 0$, therefore,

$$\lim_{k\to\infty} \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1})) = 1.$$

Since $\beta \in S$, we have

$$\lim_{k\to\infty} d(gx_{n(k)-1}, gx_{m(k)-1}) = 0. \tag{2.28}$$

and

$$\lim_{k\to\infty} d(gy_{n(k)-1}, gy_{m(k)-1}) = 0.$$

Result (2.28) contradicts to (2.25). Hence sequence $\{gx_n\}$ is a Cauchy sequence. Similarly we can prove that if $\{gx_n\}$ is a Cauchy sequence then so is $\{gy_n\}$.

Hence sequences $\{gx_n\}$ and $\{gy_n\}$ both are Cauchy sequences. Thus statements (i) and (ii) are satisfied. This proves (c).

Remark 2.1: The condition of convex on ψ in Lemma 2.1 may be replaced by affine.

Theorem 2.2 Let $(X, \leq, d)$ be a partially ordered complete metric space and $f : X \times X \to X$, $g : X \to X$ satisfy (H_1) - (H_3). Assume further that

(i) g and f are continuous,

(ii) g commutes with f.

If there exists $x_0, y_0 \in X$ such that

$$gx_0 \leq f(x_0, y_0) \text{ and } f(y_0, x_0) \leq gy_0,$$

then f and g have a coupled coincidence, that is, there exists $x, y \in X$ such that $gx = f(x, y)$ and $gy = f(y, x)$.

Proof: Choose $x_0, y_0 \in X$ such that $gx_0 \leq f(x_0, y_0)$ and $f(y_0, x_0) \leq gy_0$.

Since $f(X \times X) \subset g(X)$, we can constructed sequences $\{gx_n\}$ and $\{gy_n\}$ defined by (2.1). Using Lemma 2.1, we have

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0$$

and

$$\lim_{n\to\infty} d(gy_n, gy_{n+1}) = 0.$$

Suppose there exists an integer n_0 such that

$$d(gx_{n0}, gx_{n0+1}) = 0 \tag{2:29}$$

and

$$d(gy_{n0}, gy_{n0+1}) = 0. \tag{2:30}$$

Since g commutes with f, we have

$$g(gx_{n0}) = g(gx_{n0+1}) = g(f(x_{n0}, y_{n0})) = f(gx_{n0}, gy_{n0}),$$

and

$$g(gy_{n0}) = f(gy_{n0}, gx_{n0}).$$

Therefore, (gx_{n0}, gy_{n0}) is a point of coupled coincidence. In another case, suppose there does not exists any n_0 satisfying (3.29) and (3.30). From Lemma 2.1, we observe that sequences $\{gx_n\}$ and $\{gy_n\}$ defined by (2.1) are Cauchy sequences. Since X is complete, there exists $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \text{ and } \lim_{n\to\infty} gy_n = y.$$

The continuity of g yields

$$\lim_{n\to\infty} g(gx_n) = gx, \text{ and } \lim_{n\to\infty} g(gy_n) = gy.$$

Since g commutes with f we have

$$g(gx_{n+1}) = g(f(x_n, y_n)) = f(gx_n, gy_n) \tag{2:31}$$

and

$$g(gy_{n+1}) = g(f(y_n, x_n)) = f(gy_n, gx_n) \tag{2:32}$$

Taking limit as $n \to \infty$ in (2.31) and (2.32) and using the continuity of f, we get

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(f(x_n, y_n)) = \lim_{n\to\infty} f(gx_n, gy_n) = f(x, y)$$

and

$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(f(y_n, x_n)) = \lim_{n\to\infty} f(gy_n, gx_n) = f(y, x).$$

Thus $gx = f(x, y)$ and $gy = f(y, x)$. Therefore, g and f have a coupled coincidence.

Corollary 2.1 Let $(X, \leq, d)$ be a partially ordered complete metric space and $f : X \times X \to X$ be continuous mixed monotone mapping satisfying

$$\psi(d(f(x,y),\ f(u,v))) \leq \beta(d(x,u)+d(y,v))\psi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$

where $\beta \in S$ and ψ is convex (or affine) altering function. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq f(x_0, y_0) \text{ and } f(x_0, y_0) \leq y_0$$

then f has a unique fixed point.

Proof: Taking $gx = x$, in Theorem 2.2 and using Theorem 2.1, we obtain Corollary 2.1.

Corollary 2.2 Let $(X,\leq, d)$ be a partially ordered complete metric space and $f : X \times X \to X$ be continuous mixed monotone mapping satisfying

$$d(f(x,y),\ f(u,v)) \leq \beta(d(x,u)+d(y,v))\psi\left(\frac{d(x,u)+d(y,v)}{2}\right)$$

where $\beta \in S$. If there exists $x_0, y_0 \in X$ such that

$$x_0 \leq f(x_0, y_0) \text{ and } f(y_0, x_0) \leq y_0,$$

then f has a unique coupled fixed point.

Proof: Letting $\psi(t) = t$, in Corollary 2.1, we obtain Corollary 2.2.

The following example illustrates Corollary 2.2.

Example 2.1 Let $X = [-a, a]$ with usual partial order $\leq$ and usual metric d. Clearly $(X, \leq, d)$ is a partially ordered complete metric space. Let $\psi(t) = t$ and $\beta = 2a/2a+\alpha$, obviously ψ is convex (affine) altering function and $\beta \in S$. Define $f : X \times X \to X$ by

$$f(x,y)=\frac{x-y}{k}, k\geq 6.$$

Then f is continuous and has mixed monotone property. For $x, y, u, v \in X$, we have

$$d(f(x,y), f(u,v)=d\left(\frac{x-y}{k},\frac{u-v}{k}\right)$$

$$=\left|\frac{x-y}{k}-\frac{u-v}{k}\right|$$

$$\leq \frac{1}{k}|x-u|+\frac{1}{k}|y-v|.$$

$$d(x, u) + d(y, v)) = |x-u| + |y-v|.$$

$$\beta d(x,u)+d(y,v))=\frac{2a}{2a+d(x,u)+d(y,v)}$$

$$\geq \frac{2a}{2a+2a+2a}$$

$$=\frac{1}{3}$$

Therefore

$$\beta d((x,u)+d(u,v))\left(\frac{d(x,u)+d(y,v)}{2}\right) \geq \frac{1}{3}\cdot\frac{1}{3}(|x-u|+|y-v|)$$

$$\geq \frac{1}{k}(|x-u|+|y-v|)$$

$$\geq d(f(x,y), f(u,v)).$$

Since $k \geq 6$, we can choose infinitely many $x_0, y_0 \in X$ satisfying,

$$(k-1)x_0+y_0\leq 0 \text{ and } x_0+(k-1)y_0\geq 0 \qquad (2:33).$$

This implies that

$$x_0\leq\frac{x_0-y_0}{k} \text{ and } \frac{y_0-x_0}{k}\leq y_0.$$

Therefore,

$$x_0 \leq f(x_0, y0) \text{ and } f(y_0, x_0)\leq y_0.$$

Hence, by Corollary 2.2, f has a unique coupled fixed point.

Remark 2.2 In example 2.1, $x = 0$ is the unique fixed point of f.

Remark 2.3 For any $x_0, y_0 \in X$ satisfying (2.33), define sequences $\{x_n\}$, $\{y_n\}$ by

$$x_{n+1} = f(x_n, y_n), y_{n+1} = f(y_n, x_n).$$

Clearly we get

$$x_{n+1}=-\left(\frac{2}{k}\right)^n y_1; y_{n+1}=\left(\frac{2}{k}\right)^n y_1.$$

Since $k \geq 6$, $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$. Therefore, sequences $\{x_n\}$, $\{y_n\}$ converges to a fixed point of f.

REFERENCES

[1] R.P. Agarwal, M. A. El-Gebeily and D.O'Regan, Generalized contractions in partially ordered metric spaces, Applicable Analysis, 87(1) (2008), 109-116.

[2] G.V.R. Babu, B. Lalita and M.L. Sandhya, Common fixed point theorems involving two generalized alternating distance functions in four variables, Procedings of the Jangjeon Mathematical Society, 10(1) (2007), 83-93.

[3] G.V.R. Babu and K.P.R. Sastry, Some fixed point theorems by alternating distances between the points, Indian Journal of Pure and Applied Mathematics, 30(6)(1999), 641-647.

[4] T.G. Bhaskar and V. Laxmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis: Theory, Method and Applications, 65(7) (2006), 1379-1393.

[5] V.C. Borkar and S.T. Patil, Class of mixed monotone operators and its applications to operator equations, Bulletin of Pure and Applied Sciences, 22(E)(1)(2003), 179-187.

[6] J. Caballero, J. Harjani and K. Sadarangani, Contractive like mapping principles in ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory and Applications, (2003), Article ID 916064.

[7] Lj. B. Ciric, A generalization of Banach contraction principle, Proc. American Mathematical Society, 45(2) (1974), 267-273.

[8] Lj. B. Ciric and V. Laxmikantham, Coupled fixed point theorems for non-linear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Method and Applications, 70(12) (2009), 4341-4349.

[9] D. Gau and V. Laxmikantham, Coupled fixed points of nonlinear operators with applications, Nonlinear Analysis: Theory, Method and Applications, 11(5) (1987), 623-632.

[10] D. Gau and V. Laxmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, (1988).

[11] N. Hussain, M.H. Shah, and M.A. Kutbi, Coupled coinsidence point theorem for nonlinear contraction in partially ordered Quasi-Metric spaces with a Q-function, Fixed Point Theory and Applications, (2011), Article ID 703938.

[12] S.V.R. Naidu, Some fixed point theorems in metric spaces by alternating distances, Czechoslovak Mathematical Journal, 53(1)(2003), 205-212.

[13] J.J. Nieto, and R. Rodriguez-Lpoez, Contraction mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22(3) (2005), 223-229.

[14] Yong-Zhuo Chen, Existence theorems of coupled fixed points, J. Math. Anal. Appl., 154 (1991), 142-150.

[15] Zang Zitao, New fixed point theorems of mixed monotone operators and applications. Math. Anal. Appl., 204 (1996), 307-319.

16] J. Caballero, J. Harjani, and K. Sadarangani, Contractive-Like Mapping Principles in Ordered Metric Spaces and Application to Ordinary Differential Equations, Hindawi Publishing Corporation, Fixed Point Theory and Applications, Volume 2010, Article ID 916064.

[17] Saud M. Alsulami And Abdullah Alotaibi, Coupled Coincidence Point Theorems For Compatible Mappings In Partially Ordered Metric Spaces, Bulletin Of Mathematical Analysis And Applications, Volume 4 Issue 2 (2012), Pages 129-138.

[18] N. Hussain,1 M. H. Shah,2 and M. A. Kutbi1, Coupled Coincidence Point Theorems for Nonlinear Contractions in Partially Ordered Quasi-Metric Spaces with a *Q*-Function, Fixed Point Theory and Applications, Volume 2011, Article ID 703938, 21 pages, doi:10.1155/2011/703938