TRIPLE FIXED POINTS THEOREMS ON CONE BANACH SPACE

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Abstract: We know that every cone normed space is cone metric space. In this note, we generalize the concept of triple fixed point theorems in cone normed space obtained by Erdal Karapinar and Duran Turkoglu [1].

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Introduction: In 2007, Huang and Zhang [6] introduced the concept of cone metric space and obtained some fixed point theorems for contractive mappings in this space. The concept of triple fixed point was introduced by Beinde and Borcut [2]. H. Ayadi and E. Karapinar [4] proved the triple fixed point theorems in partially ordered metric space on another function.

E. Karapinar and Duran Turkoglu [1] proved best approximate theorem for a couple in Cone Banach Space (CNS). In this manuscript, we generalize the results of E. Karapinar and Duran Turkoglu [1].

Let E be a real Banach Space. A subset P of E is called a cone if

(i) P is closed, nonempty and P ≠ {0}
(ii) a, b ∈ R, a, b ≥ 0, and x, y ∈ P imply ax + by ∈ P
(iii) P ∩ (−P) = {0}

Given a cone P ⊂ E, we define the partial ordering ≤ with respect to P by x ≤ y if and only if y − x ∈ P. We write x < y to denote that x ≤ y but x ≠ y, while x ≪ y will stand for y − x ∈ int P (interior of P).

There are two kinds of cone. They are normal cone and non-normal cones. A cone P ⊂ E is normal if there is a number K > 0 such that for all x, y ∈ P , 0 ≤ x ≤ y ⇒ ||x|| ≤ K ||y||. In other words if xₙ ≤ yₙ ≤ zₙ and limₙ→∞ xₙ = limₙ→∞ zₙ = x imply limₙ→∞ yₙ = x.

The least positive number K satisfying ||x|| ≤ K||y|| is called the normal constant of P. It is clear that K ≥ 1.

Definition 1.1[5]: Let X be a nonempty set. Suppose the mapping d: X × X → E satisfies

i) 0 < d(x, y) for all x, y ∈ X and d(x, y) = 0 if and only if x = y;
ii) d(x, y) = d(y, x) for all x, y ∈ X;
iii) d(x, y) ≤ d(x, z) + d(z, y) for all x, y, z ∈ X.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Also, we call d as cone metric.
**Definition 1.2:** Let $(X,d)$ be a CNS and $I = [0,1]$ be the closed unit interval. A continuous mapping $W : X \times X \times I \rightarrow X$ is said to be a convex structure on $X$ if for all $x, y, z \in X$ and $t \in I$

$$\|u - W(x, y, z, t)\|_p \leq t\|u - x\|_p + (1-t)\|u - y\|_p + (1-t^2)\|u - z\|_p$$

for all $u \in X$.

A CNS together with a convex structure is said to be convex CNS. A subset $Y \subseteq X$ is convex, if $W(x, y, z, t) \in Y$ holds for $x, y, z \in X$ and $t \in I$.

**Definition 1.3:** Let $X$ be a CNS and $K$ and $C$ the nonempty convex subsets of $X$. A mapping $g : K \rightarrow X$ is said to be almost quasi-convex with respect to $C$ if

$$g((tx + (1-t)y + (1-t^2)z - w) \leq C_s([tx + (1-t)y + (1-t^2)z], w)$$

Where $C_s([tx + (1-t)y + (1-t^2)z], w) \in \{\|g(x) - w\|_p, \|g(y) - w\|_p, \|g(z) - w\|_p\}$

for all $x, y, z \in K, w \in C$ and $0 < t < 1$.

**Definition 1.4:** Let $X$ be a vector space over $\mathbb{R}$. Suppose the mapping $\| \cdot \|_p : X \rightarrow E$ satisfies the following:

1. $\|x\|_p > 0$ for all $x \in X$.
2. $\|x\|_p = 0$ if and only if $x = 0$.
3. $\|x + y\|_p \leq \|x\|_p + \|y\|_p$, for all $x, y \in X$.
4. $\|kx\|_p = |k|\|x\|_p$ for all $k \in \mathbb{R}$.

Then $\| \cdot \|_p$ is called cone norm on $X$, and the pair $(X, \| \cdot \|_p)$ is a called a cone normed space (CNS).

**Remark 1.5 [1]:** Each Cone Normed Space (CNS) is Cone Metric Space (CMS).

**Definition 1.6 [1]:** Let $(X, \| \cdot \|_p)$ be a CNS. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then

1. $\{x_n\}$ converges to $x$ whenever for every $c \in E$ with $0 << c$ there is a natural number $N$, such that $\|x_n - x\|_p << c$ for all $n \geq N$. It is denoted by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number $N$, such that $\|x_n - x_m\|_p << c$ for all $n, m \geq N$.
3. $(X, \| \cdot \|_p)$ is a complete cone normed space if every Cauchy sequence is convergent.

**Lemma 1.7 [1]:** Let $(X, \| \cdot \|_p)$ be a CNS. Let $P$ be normal cone with normal constant $K$, and let $\{x_n\}$ be sequence in $X$. Then

1. The sequence $\{x_n\}$ converges to $x$ if and only if $\|x_n - x\|_p \rightarrow 0$ as $n \rightarrow \infty$.
2. The sequence \( \{ x_n \} \) is Cauchy if and only if \( \| x_n - x_m \|_p \to 0 \), \( n, m \to \infty \).

3. The sequence \( \{ x_n \} \) converges to \( X \) and the sequence \( \{ y_n \} \) converges to \( Y \) and then \( \| x_n - y_n \|_p \to \| x - y \|_p \).

2. Triple sequence:

Let \((X, d)\) be a cone metric space and \( X^3 = X \times X \times X \). Then the mapping \( \phi: X^3 \times X^3 \times X^3 \to E \) such that \( \phi((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)) = d(x_1, x_2) + d(x_2, y_2) + d(x_3, y_3) \) for all \((x_i, y_i, z_i) \in X^3\), \( i = 1, 2, 3 \). A sequence \((\{x_n\}, \{y_n\}, \{z_n\}) \in X^3 \) is said to be a triple sequence of \( X \). A sequence \((\{x_n\}, \{y_n\}, \{z_n\}) \in X^3 \) is said to be convergent to \((x, y, z) \in X^3 \) if for every \( c \in \text{int}(P) \), there exists a natural number \( M > 0 \) such that \( \phi((x_n, y_n, z_n), (x, y, z)) < c \) for all \( n > M \).

**Lemma 2.1:** Let \( w_n = (x_n, y_n, z_n) \in X^3 \) and \( w = (x, y, z) \in X^3 \). Then \( w_n \to w \) if and only if \( x_n \to x \), \( y_n \to y \) and \( z_n \to z \).

**Proof:** Let us assume that \( w_n \to w \). Then, for any \( c \in \text{int}(P) \), there exists \( M > 0 \) such that \( \phi((x_n, y_n, z_n), (x, y, z)) = d(x_n, x) + d(y_n, y) + d(z_n, z) < c \) for all \( n > M \). Hence \( d(x_n, x) < c \), \( d(y_n, y) < c \) and \( d(z_n, z) < c \) for all \( n > M \), that is, \( x_n \to x \), \( y_n \to y \) and \( z_n \to z \).

Conversely, let us assume that \( x_n \to x \), \( y_n \to y \) and \( z_n \to z \). Thus for any \( c \in \text{int}(P) \), there exists \( M_1, M_2, M_3 > 0 \) such that \( d(x_n, x) < \frac{c}{3} \) for all \( n > M_1 \), \( d(y_n, y) < \frac{c}{3} \) for all \( n > M_2 \) and \( d(z_n, z) < \frac{c}{3} \) for all \( n > M_3 \). Hence \( \phi((x_n, y_n, z_n), (x, y, z)) = d(x_n, x) + d(y_n, y) + d(z_n, z) < c \) for all \( n > M \), where \( M = \max\{M_1, M_2, M_3\} \).

**Definition 2.2:** Let \((X, d)\) be a cone metric space. A function \( \phi: X \times X \times X \to X \) is sequentially continuous if \( \phi((x_n, y_n, z_n), (x, y, z)) \to 0 \) implies that \( d(\phi(x_n, y_n, z_n), \phi(x, y, z)) \to 0 \).

**Lemma 2.3:** Let \((X, d)\) be a cone metric space. Then \( \phi: (X, d) \to (X, d) \) is continuous if and only if \( \phi \) is sequentially continuous.

**Definition 2.4:** Let \((X, \leq)\) be partially ordered set and \( \phi: X \times X \times X \to X \). Then \( \phi \) is said to have mixed monotone property if \( \phi(x, y, z) \) is monotone non-decreasing in \( x \), \( y \) and \( z \), that is, any \( x, y, z \in X \) such that

\[
x_1 \leq x_2 \Rightarrow \phi(x_1, y, z) \leq \phi(x_2, y, z)
\]

for all \( x_1, x_2 \in X \)

\[
y_1 \leq y_2 \Rightarrow \phi(x, y_1, z) \leq \phi(x, y_2, z)
\]

for all \( y_1, y_2 \in X \)

\[
z_1 \leq z_2 \Rightarrow \phi(x, y, z_1) \leq \phi(x, y, z_2)
\]

for all \( z_1, z_2 \in X \).

This definition reduces the notion of mixed monotone function on \( R^3 \) where \( \leq \) denotes usual total order in \( R^3 \).

**Definition 2.5:** An element \((x, y, z) \in X \times X \times X \) is said to be a triple fixed point of \( \phi: X \times X \times X \to X \) if \( \phi(x, y, z) = x, \phi(y, z, x) = y \) & \( \phi(z, x, y) = z \).
Throughout this paper, let \((X, \leq)\) be partially ordered set and let \(d\) be a cone metric on \(X\) such that \((X, d)\) is a complete cone metric space over the normal cone \(P\) with normal constant \(K\). Also the space \(R^3\) satisfy the following
\[
(u, v, w) \leq (x, y, z)
\]
\[
\Rightarrow u \leq x, v \leq y \text{ and } w \leq z \text{ for all } ((u, v, w), (x, y, z)) \in X^3.
\]

**Definition 2.6:** Let \((X, d)\) be a cone metric space and \(A \subset X\). A set \(A\) is said to be sequentially compact if for any sequence \([x_n]\) in \(A\) there is a subsequence \([x_{n_k}]\) of \([x_n]\) such that \([x_{n_k}]\) is convergent in \(A\).

**Remark 2.7 [1]:** Every cone metric space \((X, d)\) is a topological space which is dented by \((X, \tau_\infty)\). Moreover, a subset \(A \subset X\) is sequentially compact if and only if \(A\) is compact.

**Definition 2.8:** Let \(K\) be a non-empty subset of a cone metric space \((X, d)\). A set-valued mapping \(H : K \rightarrow 2^X\) is called KKM map if for every finite subset \(\{x_1, x_2, x_3, ..., x_n\}\) of \(K\), \(Co\{x_1, x_2, x_3, ..., x_n\} \subset \bigcup_{i=1}^{n} H(x_i)\) where \(Co\) denotes the convex hull.

**Lemma 2.9:** Let \(X\) be a topological vector space, let \(K\) be a nonempty subset of \(X\) and \(H : K \rightarrow 2^X\) be a KKM map with closed values. If \(H(x)\) is compact for at least one \(x \in K\), then \(\bigcap_{x \in K} H(x) \neq \emptyset\).

**Theorem 2.10:** Let \((X, \| \cdot \|_p)\) be a cone metric space over strongly mini-dhedral cone \(P\), and let \(K\) be a nonempty convex compact subset of \(X\). If \(f : K \times K \times K \rightarrow X\) is continuous mapping and \(g : K \rightarrow X\) is continuous almost quasi-convex mapping with respect to \(f\), then there exists \((x_0, y_0, z_0) \in K \times K \times K\) such that
\[
\inf_{(x,y,z) \in K \times K \times K} \{\|g(x) - f(x, y, z)\|_p + \|g(y) - f(x, y, z)\|_p + \|g(z) - f(x, y, z)\|_p\}.
\]

**Proof:** Let \(H : K \times K \times K \rightarrow 2^{K \times K \times K}\) be defined by
\[
H(u, v, w) = \{ (x, y, z) \in K \times K \times K : \|g(x) - f(x_0, y_0, z_0)\|_p + \|g(y) - f(x_0, y_0, z_0)\|_p + \|g(z) - f(x_0, y_0, z_0)\|_p \}
\]
for each \((u, v, w) \in K \times K\). Since, \((u, v, w) \in H(u, v, w)\) then \(H(u, v, w) \neq \emptyset\). Since \(f\) and \(g\) are continuous therefore, \(H(u, v, w)\) is closed for each \((u, v, w)\). Since \(K\) is compact, then \(H(u, v, w)\) is compact for each \((u, v, w)\). Thus, \(H\) is a KKM map.

Let \((u_i, v_j, w_l) \in K \times K \times K\), \(i \in I, j \in J \& l \in L\) where \(I, J, L\) are finite subsets of \(N\). Then there exists \((u_0, v_0, w_0) \in Co\{(u_i, v_j, w_l) : (i, j, l) \in I \times J \times L\}\), so that
\[
(u_0, v_0, w_0) \notin \{H(u_i, v_j, w_l) : (i, j, l) \in I \times J \times L\}.
\]

Then \(\exists t_{i,j,l} \geq 0, (i, j, l) \in I \times J \times L\) such that \((u_0, v_0, w_0) = \sum_{(i,j,l) \in I \times J \times L} t_{i,j,l} (u_i, v_j, w_l)\) and \(\sum_{(i,j,l) \in I \times J \times L} t_{i,j,l} = 1\).
Set \( t_i = \sum_{(j,l) \in J \times L} t_{ijl}, z_j = \sum_{(i,l) \in I \times L} t_{ijl} \) and \( q_l = \sum_{(i,j) \in I \times J} t_{ijl} \). Then, \( \sum_{(j,l) \in J \times L} t_i = 1 \), \( \sum_{(i,l) \in I \times L} z_j = 1 \) and \( \sum_{(i,j) \in I \times J} q_l = 1 \) and 
\[
\sum_{(j,l) \in J \times L} t_i u_j = u_0, \sum_{(i,l) \in I \times L} z_j v_j = v_0 \text{ and } \sum_{(i,j) \in I \times J} q_l w_j = w_0.
\]

Also, \( g \) is almost quasi-convex with respect to \( f : K \times K \times K \to X \) gives
\[
\left\| g(u_0) - f(u_0, v_0, w_0) \right\|_p \leq C_g(u_0, f(u_0, v_0, w_0))
\]
\[
\left\| g(v_0) - f(v_0, w_0, u_0) \right\|_p \leq C_g(v_0, f(v_0, w_0, u_0))
\]
And 
\[
\left\| g(w_0) - f(w_0, u_0, v_0) \right\|_p \leq C_g(w_0, f(w_0, u_0, v_0))
\]

Where \( C_g(u_0, f(u_0, v_0, w_0)) \in \{ \left\| g(u_i) - f(u_0, v_0, w_0) \right\|_p : i \in I \} \)
\[
C_g(v_0, f(v_0, w_0, u_0)) \in \{ \left\| g(v_j) - f(v_0, w_0, u_0) \right\|_p : j \in J \}
\]
And \( C_g(w_0, f(w_0, u_0, v_0)) \in \{ \left\| g(w_l) - f(w_0, u_0, v_0) \right\|_p : l \in L \} \)

Thus, 
\[
\left\| g(u_0) - f(u_0, v_0, w_0) \right\|_p + \left\| g(v_0) - f(v_0, w_0, u_0) \right\|_p + \left\| g(w_0) - f(w_0, u_0, v_0) \right\|_p
\]
\[
\leq \inf \{ \left\| g(u_i) - f(u_0, v_0, w_0) \right\|_p : i \in I \} + \inf \{ \left\| g(v_j) - f(v_0, w_0, u_0) \right\|_p : j \in J \} + \inf \{ \left\| g(w_l) - f(w_0, u_0, v_0) \right\|_p : l \in L \}
\]

Therefore, 
\[
\left\| g(u_0) - f(u_0, v_0, w_0) \right\|_p + \left\| g(v_0) - f(v_0, w_0, u_0) \right\|_p + \left\| g(w_0) - f(w_0, u_0, v_0) \right\|_p
\]
\[
> \left\| g(u_0) - f(u_0, v_0, w_0) \right\|_p + \left\| g(v_0) - f(v_0, w_0, u_0) \right\|_p + \left\| g(w_0) - f(w_0, u_0, v_0) \right\|_p
\]
For all \((i, j, l) \in I \times J \times L\). This is a contradiction. Hence \( H \) is KKM mapping. If follows that there exists 
\((x_0, y_0, z_0) \in K \times K \times K\) such that 
\((x_0, y_0, z_0) \in H(x, y, z)\) for all \((x, y, z) \in K \times K \times K\). Thus
\[
\left\| g(x_0) - f(x_0, y_0, z_0) \right\|_p + \left\| g(y_0) - f(y_0, z_0, x_0) \right\|_p + \left\| g(z_0) - f(z_0, x_0, y_0) \right\|_p
\]
\[
\leq \left\| g(x) - f(x, y_0, z_0) \right\|_p + \left\| g(y) - f(y_0, z_0, x_0) \right\|_p + \left\| g(z) - f(z_0, x_0, y_0) \right\|_p
\]
For all \((x, y, z) \in K \times K \times K\).

**Theorem 2.11:** Let \((X, \left\| . \right\|_p)\) be a cone metric space over strongly minidhedral cone \( P \) and let \( K \) be nonempty convex compact subset of \( X \). If \( f : K^3 \to X \) is continuous mapping and \( g : X \to X \) is continuous almost quasi-convex mapping with respect to \( f \) such that \( f(K \times K \times K) \subseteq g(K) \), then \( f \) and \( g \) have a coupled coincidence point.

**Proof:** By the above theorem, there exists \((x_0, y_0, z_0) \in K \times K \times K\) such that
\[ \|g(x_0) - f(x_0, y_0, z_0)\|_p + \|g(y_0) - f(x_0, y_0, z_0)\|_p + \|g(z_0) - f(x_0, y_0, z_0)\|_p = \inf_{(x,y,z) \in K \times K \times K} \left\{ \|g(x) - f(x_0, y_0, z_0)\|_p + \|g(y) - f(y_0, z_0, x_0)\|_p + \|g(z) - f(z_0, x_0, y_0)\|_p \right\} \]

Since \( f(K \times K \times K) \subset g(K) \), therefore
\[ = \inf_{(x,y,z) \in K \times K \times K} \left\{ \|g(x) - f(x_0, y_0, z_0)\|_p + \|g(y) - f(y_0, z_0, x_0)\|_p + \|g(z) - f(z_0, x_0, y_0)\|_p \right\} = 0 \]

Then
\[ \|g(x_0) - f(x_0, y_0, z_0)\|_p + \|g(y_0) - f(x_0, y_0, z_0)\|_p + \|g(z_0) - f(x_0, y_0, z_0)\|_p = 0 \]

Therefore,
\[ g(x_0) = f(x_0, y_0, z_0), \ g(y_0) = f(y_0, z_0, x_0) \text{ and } g(z_0) = f(x_0, y_0, z_0). \]

Taking \( g : K \to X \) as an identity, \( g(x) = x \). This completes the proof.

**Theorem 2.12:** Let \((X, \|\cdot\|_p)\) be a cone metric space over strongly minidhedral cone \(P\) and let \(K\) be nonempty convex compact subset of \(X\). If \( f : K^3 \to K \) is continuous mapping, then \( f \) has a coupled fixed point.

**Theorem 2.13:** Let \((X, \|\cdot\|_p)\) be a cone metric space over strongly minidhedral cone \(P\) and let \(K\) be nonempty convex compact subset of \(X\). If \( f : K^3 \to X \) is continuous mapping, then either \( f \) has a triple fixed point or there exists \((x_0, y_0, z_0) \in (\partial K \times K \cup K \times K \cup K \times K)\) such that
\[ 0 < \|x_0 - f(x_0, y_0, z_0)\|_p + \|y_0 - f(x_0, y_0, z_0)\|_p + \|z_0 - f(x_0, y_0, z_0)\|_p \]
\[ \leq \|x - f(x, y, z, 0)\|_p + \|y - f(x, y, z, 0)\|_p + \|z - f(x, y, z, 0)\|_p \quad \text{......(1)} \]

For all \((x, y, z) \in K \times K \times K\).

**Proof:** If \( f \) has a triple fixed point, then it is obvious. Let us assume that \( f \) has no triple fixed points. Then there exists \((x_0, y_0, z_0) \in K \times K \times K\) such that
\[ \|g(x_0) - f(x_0, y_0, z_0)\|_p + \|g(y_0) - f(x_0, y_0, z_0)\|_p + \|g(z_0) - f(x_0, y_0, z_0)\|_p = \inf_{(x,y,z) \in K \times K \times K} \left\{ \|g(x) - f(x_0, y_0, z_0)\|_p + \|g(y) - f(y_0, z_0, x_0)\|_p + \|g(z) - f(z_0, x_0, y_0)\|_p \right\} \]

We take \( g(x) = x \) that implies (1). We are to prove that \((x_0, y_0, z_0) \in (\partial K \times K \cup K \times K \cup K \times K)\).

Then (1) implies that either \( f(x_0, y_0, z_0) \notin K \) or \( f(y_0, z_0, x_0) \notin K \) or \( f(z_0, x_0, y_0) \notin K \).

Thus \( \|x - f(x_0, y_0, z_0)\|_p = \ell \|x_0 - f(x_0, y_0, z_0)\|_p \) and
\[
\inf_{x \in K} \| x - f(x_0, y_0, z_0) \| _p \leq t \| x_0 - f(x_0, y_0, z_0) \| _p < \| x_0 - f(x_0, y_0, z_0) \| _p
\]

This is a contradiction. Similarly, we can obtain \( f(y_0, z_0, x_0) \not\in K \) and \( f(z_0, x_0, y_0) \not\in K \).

Hence \( (x_0, y_0, z_0) \in (\partial K \times K \cup K \times K \cup K \times \partial K) \)

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