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SUPER CONNECTED DOMINATION IN GRAPHS

Bea Elaine P. Fedellaga[1]

Department of Mathematics,
University of Eastern Philippines, 6400 Catarman, Philippines
[1]email: beaelainefedellaga@gmail.com

Enrico L. Enriquez[2], Carmelita M. Loquias[3], Grace M. Estrada[4], Margie L. Baterna[5]

Department of Mathematics
School of Arts and Sciences
University of San Carlos, 6000 Cebu City, Philippines
[2]email: enricolenriquez@yahoo.com
[3]email: cmloquias@usc.edu.ph
[4]email: gmestrada@usc.edu.ph
[5]email: mlbaterna@usc.edu.ph

Abstract: In this paper, we initiate the study of super connected dominating set of a graph G by giving the super connected domination number of some special graphs. Further, we shows that given positive integers k, m and n such that $n \geq 2$ and $1 \leq k \leq m \leq n-1$, there exists a connected graph G with $|V(G)| = n$, $\gamma_c(G) = k$, and $\gamma_{spc}(G) = m$. Finally, we characterize the super connected dominating set of the join, corona, and Cartesian product of two graphs.

Mathematics Subject Classification: 05C69

Keywords:dominating set, connected dominating set, super dominating set, super connected dominating set

1 INTRODUCTION

Let G be a simple graph. A subset S of a vertex set $V(G)$ is a dominating set of G if for every vertex $v \in V(G) \setminus S$, there exists a vertex $x \in S$ such that xv is an edge of G. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Dominating sets have several applications in a variety of fields, including communication and electrical networks, protection and location strategies, data structures and others. For more background on dominating sets, the reader may refer to [1,2,3,4]. Some variants of domination in graphs are found in [5,6,7,8,9,10,11]. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [12].

A graph G is connected if there is at least one path that connects every two vertices $x, y \in V(G)$, otherwise, G is disconnected. A dominating set S is said to be connected dominating set (CDS), if the induced subgraph $\langle S \rangle$ is connected. The connected domination number of G is the minimum cardinality of a connected dominating set in G and is denoted by $\gamma_c(G)$.Sampathkumar and Walikar (1979) defined connected domination in graphs in the paper entitled " The connected domination number of a graph".

Domination in graphs has several parameters that have significant contributions in graph theory, these include super domination and connected domination in graphs. The super dominating sets in graphs was initiated by Lemanska et.al. [13]. Variation of super domination in graphs can be read in [14, 15, 16]. A set $D \subset V(G)$ is called a super dominating set if for every vertex $u \in V(G) \setminus D$, there exists $v \in D$ such that $N_G(v) \cap (V(G) \setminus D) = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating set in G denoted by $\gamma_{sp}(G)$.

Motivated by super domination and connected domination in graphs , we initiate the study of super connected domination in graphs. A connected dominating set $S \subseteq V(G)$ is a super connected dominating set if for every $u \in V(G) \setminus S$, there exists $v \in$

S such that $N_G(v) \cap (V(G) \setminus S) = \{u\}$. The minimum cardinality of a super connected dominating set of G, denoted by $\gamma_{spc}(G)$, is called the super connected domination number of G. For general concepts we refer the reader to [17].

2 RESULTS

This section presents some results on super connected domination in the join, corona, and cartesian product of two graphs.

Remark 2.1*A super connected dominating set is a super dominating and a connected dominating set of a nontrivial graph G.*

The following result says that the value of the parameter$\gamma_{spc}(G)$ranges over all positive integers, $1,2,\ldots,n-1$.

Theorem 2.2*Given positive integers k, m and n such that $n \geq 2$ and $1 \leq k \leq m \leq n-1$, there exists a connected graph G with $|V(G)| = n, \gamma_c(G) = k$, and $\gamma_{spc}(G) = m$.*

Proof: Consider the following cases:

Case 1. Suppose $m = n-1$.

Let $G = K_n$. (see Figure 5). Then, clearly, $|V(G)| = n$ and $\gamma_c(G) = 1 = k$ and $\gamma_{spc}(G) = n-1 = m$.

Case 2. Suppose $m < n-1$.

Consider $1 \leq k = m$. Let $G = P_k \circ K_1$. (see Figure 6). Then the set $S = V(P_k)$ is a connected dominating set and a super dominating set of G. Since S is both minimum connected and super dominating sets, it follows that S is a minimum super connected dominating set of G. Thus, $|V(G)| = 2k = n, \gamma_c(G) = |S| = k$, and $\gamma_{spc}(G) = k = m$.

Consider $1 < k < m$. Let $G = P_3 \square P_k$ where $k \geq 1, m = 2k$, and $n = 3k$. Let $V(P_3) = \{x_1, x_2, x_3\}$ and $E(P_3) = \{x_1x_2, x_2x_3\}, V(P_k) = \{y_1, y_2, \ldots, y_k\}$ and $E(P_k) = \{y_1y_2, y_2y_3, \ldots, y_{k-1}y_k\}$. (see Figure 7).The set $A = \{(x_2, y_r): r = 1,2,\ldots,k\}$ is the minimum connected dominating set of G and $B = A \cup \backslash\{(x_3, y_r): r = 1,2,\ldots,k\}$ is the minimum super connected dominating set of G. Thus, $|V(G)| = |P_3 \square P_k| = 3k = n, \gamma_c(G) = |A| = k$, and $\gamma_{spc}(G) = |B| = k + k = m$.

Consider $1 = k < m$. Let $G = P_3 + P_r$ where $r \equiv 0(mod\ 4)$, $n = r + 3$ and $2m = n + 3$. Let $V(P_3) = \{x_1, x_2, x_3\}$ and $E(P_3) = \{x_1x_2, x_2x_3\}, V(P_r) = \{y_1, y_2, \ldots, y_r\}$ and $E(P_r) = \{y_1y_2, y_2y_3, \ldots, y_{r-1}y_r\}$. (see Figure 9). The set $A = \{x_2\}$ is the minimum connected dominating set of G and $B = V(P_3) \cup \left\{y_{4s-2}: s = 1,2,\ldots,\frac{r}{4}\right\} \cup \{y_{4s-1}: s = 1,2,\ldots r/4\}$ is the minimum super connected dominating set of G. Thus, $|V(G)| = |P_3 + P_r| = 3 + r = n, \gamma_c(G) = |A| = 1 = k$, and $\gamma_{spc}(G) = |V(P_3)| + r/4 + r/4 = 3 + r/2 = m$. This proves the assertion. □

The next result is an immediate consequence of Theorem 2.2.

Corollary 2.3*The difference $\gamma_{spc}(G) - \gamma_c(G)$ can be made arbitrarily large.*

We need the following results for the characterization of the super connected domination in some binary operations such as join, corona, lexicographic and cartesian product.

The following lemma is used in the characterization of super connected dominating set in the join of two graphs.

Lemma 2.4*Let G and H be connected non-complete graphs. If $S_G = V(G) \setminus \{a\}$ for some $a \in V(G), S_H = V(H) \setminus \{b\}$ for some $b \in V(H)$, then $S = S_G \cup S_H$ is a super connected dominating set of $G + H$.*

Proof: Suppose that $S_G = V(G) \setminus \{a\}$ for some $a \in V(G)$ and $S_H = V(H) \setminus \{b\}$ for some $b \in V(H)$. Then

$$S = S_G \cup S_H = (V(G) \setminus \{a\}) \cup (V(H) \setminus \{b\})$$

= (

$$= V(G+H) \setminus \{a, b\}$$

Thus, $V(G+H) \setminus S = \{a, b\}$. Since G is non-complete, choose $a \in V(G)$ such that $ac \notin E(G)$ for some $c \in V(G) \setminus \{a\} = S_G$. Similarly, since H is non-complete, choose $b \in V(H)$ such that $bd \notin E(H)$ for some $d \in V(H) \setminus \{b\} = S_H$. Consider $a \in V(G+H) \setminus S$. Then there exists $d \in S$ such that $N_{G+H}(d) \cap (V(G+H) \setminus S) = \{a\}$. Consider $b \in V(G+H) \setminus S$. Then there exists $c \in S$ such that \$$N_{G+H}(c) \cap (V(G+H) \setminus S) = \{b\}$. Thus, S is a super dominating set of $G + H$. Further, since G and H are connected non-complete graphs, $\langle S_G \rangle = \langle V(G) \setminus \{a\} \rangle$ for some $a \in V(G)$ and $\langle S_H \rangle = \langle V(H) \setminus \{b\} \rangle$ for some

$b \in V(H)$ are connected subgraphs of G and H respectively. Thus $\langle S \rangle = \langle S_G \cup S_H \rangle$ is a connected subgraph of $G + H$. Accordingly, S is a super connected dominating set of $G + H$. □

The next result is the characterization of super connected dominating set in the join of two graphs.

Theorem 2.5*Let G and H be connected non-complete graphs. Then $S = S_G \cup S_H$ is a super connected dominating set of $G + H$ where $S_G \subseteq V(G)$and $S_H \subseteq V(H)$if and only if one of the following statements is satisfied:*

(i) S_G *is a super dominating set of G and* $S_H = V(H)$.
(ii) S_H *is a super dominating set of H and* $S_G = V(G)$.
(iii) $S_G = V(G) \setminus \{w\}$ *for some* $w \in V(G), S_H = V(H) \setminus \{z\}$ *for some* $z \in V(H)$.

Proof: Suppose *that* $S = S_G \cup S_H \subseteq V(G + H)$ is a super connected dominating set of $G + H$. First, assume that $S_H = V(H)$. Suppose that to the contrary S_G is not a super dominating set of G. Then there exists $u \in V(G) \setminus S_G$ such that for all $v \in S_G, N_{G(v)} \cup (V(G) \setminus S_G) \neq \{u\}$. Thus, there exists $u \in V(G + H) \setminus S$ such that for all $v \in S, N_{G+H}(v) \cap (V(G + H) \setminus S) \neq \{u\}$, contrary to our assumption that S is a super dominating set of $G + H$. Therefore, $S_G$must be a super dominating set of G. This proves statement (i). Similarly, if we assume that $S_G = V(G)$, then statement (ii) holds. Next, assume that S_H ≠ $V(H)$and $S_G \neq V(G)$. Let $x \in V(H) \setminus S_H$ and $u \in V(G) \setminus S_G$. If we assume that there exists $u' \in V(G) \setminus S_G$ distinct from u, then $u', u \in N_{G+H}(z)$ for all $z \in S_H$. Thus, $N_{G+H}(z) \cap (V(G + H) \setminus S) = \{u, u' : u' \in V(G) \setminus S_G\} \cup \{x \in V(H) \setminus S_H : x \in N_H(z)\}$ contrary to our assumption that S is a super connected dominating set of $G + H$. This means that $V(G) \setminus S_G$ is a singleton, say $\{w\}$, and so $S_G = V(G) \setminus \{w\}$for some $w \in V(G)$. Further, if we assume that there exists $x' \in V(H) \setminus S_H$ distinct from x, then $x', x \in N_{G+H}(v)$ for all $v \in S_G$. Thus, $N_{G+H}(v) \cap (V(G + H) \setminus S) = \{x, x' : x' \in V(H) \setminus S_H\} \cup \{u \in V(G) \setminus S_G : u \in N_G(v)\}$ contrary to our assumption that S is a super connected dominating set of $G + H$. This eans that $V(H) \setminus S_H$ is a singleton, say $\{z\}$, and so $S_H = V(H) \setminus \{z\}$ for some $z \in V(H)$. This proves statement (iii).

For the converse, suppose that statement (i) is satisfied. Let $S = S_G \cup S_H$ *where* $S_G \subseteq V(G)$*and* $S_H \subseteq V(H)$.
Let $u \in V(G) \setminus S_G$. Since S_G is a super dominating set of G, there exists $v \in S_G$ *such that* $N_G(v) \cap (V(G) \setminus S_G) = \{u\}$. Since $S_H = V(H), u \in V(G) \setminus S_G \subset (V(G) \cup V(H)) \setminus (S_G \cup V(H)) = V(G + H) \setminus S$. Thus, for all $u \in V(G + H) \setminus S$, there exists $v \in S$ such that $N_{G+H}(v) \cap (V(G + H) \setminus S) = \{u\}$, that is, S is a super dominating set of $G + H$. Now, for all $v \in S_G, vx \in E(G + H)$ for all $x \in S_H = V(H)$. This implies that $S = S_G \cup S_H$ is connected. Accordingly $S = S_G \cup S_H \subseteq V(G + H)$ is a super connected dominating set of $G + H$.Similarly, if statement (ii) is satisfied, then $S = S_G \cup S_H \subseteq V(G + H)$ is a super connected dominating set of $G + H$.

Now, suppose that statement (iii) is satisfied. Then by Lemma 2.4, S is a super connected dominating set of $G + H$.This complete the proofs. □

As a consequence of Theorem 2.5, we obtain the following result.

Corollary 2.6*Let G and H be connected non-complete graphs of order m and n respectively. Then*

$$\gamma_{spc}(G + H) = min\{\gamma_{sp}(G) + n, \gamma_{sp}(H) + m, m + n - 2\}.$$

Proof: Let G and H be connected non-complete graphs of order m and n respectively. Suppose $S = S_G \cup S_H$ is a super connected dominating set of $G + H$, where $S_G \subseteq V(G)$*and* $S_H \subseteq V(H)$. Then by Theorem 2.5, statement (i) or (ii) or (iii) holds.

For statement (i), $\gamma_{spc}(G + H) \leq |S| = |S_G \cup V(H)| = |S_G| + |V(H)|$ for all super dominating set S_G of G. Thus, $\gamma_{spc}(G +$ *H≤γsp(G)+n.*

For statement (ii), $\gamma_{spc}(G + H) \leq |S| = |S_H \cup V(G)| = |S_H| + |V(G)|$ for all super dominating set $S_H$of H. Thus, $\gamma_{spc}(G +$ *H≤γsp(H)+m.*

For statement (iii), $\gamma_{spc}(G + H) \leq |S| = |S_G \cup S_H|$*where* $S_G = V(G) \setminus \{w\}$*and* $S_H = V(H) \setminus \{z\}$ for some $w \in V(G)$ and $z \in V(H)$. Thus,

$$\gamma_{spc}(G + H) \leq |(V(G) \setminus \{w\}) \cup (V(H) \setminus \{z\})|$$
$= (|(V(G)| - |\{w\}|) + (|V(H)| - |\{z\}||$
$= (m - 1) + (n - 1)$

This implies that $\gamma_{spc}(G + H) \leq min\{\gamma_{sp}(G) + n, \gamma_{sp}(H) + m, m + n - 2)\}$ as inequality (1).

Suppose S^o is a minimum super connected dominating set of $G + H$.Then

$\gamma_{spc}(G+H) = |S^o| = |S_G \cup V(H)| = |S_G| + |V(H)| \geq \gamma_{sp}(G) + n\}$for statement(i) or
$\gamma_{spc}(G+H) = |S^o| = |S_H \cup V(G)| = |S_H| + |V(G)| \geq \gamma_{sp}(H) + m$for statement (ii)

Otherwise, statement (iii) is the minimum super connected dominating set of $G+H$, that is,

$$\begin{aligned}\gamma_{spc}(G+H) &= |S^o| \\ &= |(V(G)\setminus\{w\}) \cup (V(H)\setminus\{z\}| \\ &= (|(V(G)|-|\{w\}|) + (V(H) - |\{z\}|) \\ &= (m-1)+(n-1)\end{aligned}$$

that is,

$$\gamma_{spc}(G+H) \geq (m-1)+(n-1)$$

This implies that $\gamma_{spc}(G+H) \geq \min\{\gamma_{sp}(G)+n, \gamma_{sp}(H)+m, m+n-2)\}$ as inequality (2).By combining inequality (1) and (2) we obtain the desired results.□

The following remark will be used in the characterization of super connected dominating set in the corona of two graphs.

Remark 2.7*Let G and H be nontrivial connected graphs. Then $V(G)$ is a dominating set of $G \circ H$.*

The next result is the characterization of super connected dominating set in the corona of two graphs.

Theorem 2.8*Let G and H be nontrivial connected graphs. Then a nonempty subset S of $V(G\circ H)$ is a super connected dominating set of $G \circ H$ if and only if one of the following statements is satisfied:*

(i) $S = V(G) \cup (\bigcup_{v\in V(G)} S_v)$, where S_v is a super dominating set ofH^v for all $v \in V(G)$.
(ii) $S = V(G) \cup (\bigcup_{v\in S_G} S_v) \cup (\bigcup_{x\in V(G)\setminus S_G} V(H^x))$where$S_G \subset V(G), S_G \neq \emptyset$,and S_v is a super dominating set of $H^v$for all $v \in S_G$

Proof: Suppose that a nonempty subset S of $V(G\circ H)$ is a super connected dominating set of $G\circ H$. Let S_v be a super dominating set of H^v. Then $\{v\} \cup S_v$ is a super dominating set of $v + H^v$ for all $v \in V(G)$. Since $vx \in E(v + H^v)$ for all $x \in S_v$,$\{v\}\cup S_v$ is a super connected dominating set of $v + H^v$. This implies that $\bigcup_{v\in V(G)}(\{v\}\cup S_v)$is a super connected dominating set of $G\circ H$. Let $S = \bigcup_{v\in V(G)}(\{v\}\cup S_v)$ Then $S = V(G)\cup(\bigcup_{v\in V(G)} S_v)$, where S_v is a super dominating set of H^v for all $v \in V(G)$. This proves statement (i).

Similarly, if $S_G \subset V(G), S_G \neq \emptyset$, and S_v is a super dominating set of H^v for all $v \in S_G$, then $S = V(G) \cup (\bigcup_{v\in S_G} S_v) \cup (\bigcup_{x\in V(G)\setminus S_G} V(H^x))$ is a super connected dominating set of $G\circ H$ proving statement (ii).
For the converse, suppose that statement (i) is satisfied.Then $S = \bigcup_{v\in V(G)}(\{v\}\cup S_v)$, where S_v is a super dominating set of H^v for all $v \in V(G)$. Since $V(G)$ is a dominating set of $G\circ H$ by Remark 2.7, S is a dominating set of $G\circ H$. Let $z \in S_v$. Then there exists $v \in V(G)$ such that $zv \in E(G\circ H)$. This implies that$V(G)\cup(\bigcup_{v\in V(G)} S_v)$ is connected and so S is a connected dominating set of $V(G\circ H)$. Let $u \in V(G\circ H)\setminus S$. Then $u \notin S$ and so $u \notin S_a$ for some $a \in V(G)$. Since $S_a$is a super dominating set of H^a, for each $u \in V(H^a)\setminus S_a$ there exists $x \in S_a$ such that $N_{H^a}(x) \cap (V(H^a)\setminus S_a) = \{u\}$. Since the argument is valid for all $v \in V(G)$, it follows that for each $u \in V(G\circ H)\setminus S$ there exists $v \in V(G)$ such that $N_{G\circ H}(v) \cap (V(G\circ H)\setminus S) = \{u\}$. Hence, $S$is a super dominating set of $G\circ H$. Since S is also connected, S is a super connected dominating set of $G\circ H$ by Remark 2.7.

Suppose that statement (ii) is satisfied. Then $S = V(G) \cup (\bigcup_{v\in S_G} S_v) \cup (\bigcup_{x\in V(G)\setminus S_G} V(H^x))$ where $S_G \subset V(G), S_G \neq \emptyset$, and S_v is a super dominating set of $H^v$for *all* $v \in S_G$ Using the same arguments in (i), S is a connected dominating set and a super dominating set of $G\circ H$. Hence S is a super connected dominating set of $G\circ H$.□

As a consequence of Theorem 2.8, we obtain the following result.

Corollary 2.9 *Let G and H be nontrivial connected graphs. Then $\gamma_{spc}(G\circ H) = |V(G)|(1+\gamma_{sp}(H))$.*

Proof: Suppose S is a super connected dominating set of $G\circ H$.Then $S = V(G)\cup(\bigcup_{v\in V(G)} S_v)$, where S_v is a super dominating set ofH^v for all $v \in V(G)$by Theorem 2.8. Then we have,

$$\gamma_{spc}(G\circ H) \leq |S| = |V(G) \cup (\bigcup_{v\in V(G)} S_v)|$$

$$= |V(G)| + \sum_{v\in V(G)} |S_v| , = |V(G)| + |V(G)||S_v| = |V(G)|(1 + |S_v|)$$

$\forall$ super dominating set$S_v \subset V(H^v)$. Thus,$\gamma_{spc}(G \circ H) \leq |V(G)|(1 + \gamma_{sp}(H^v))$

Suppose that S^o be a minimum super connected dominating set of $G \circ H$.Then

$$\gamma_{spc}(G \circ H) = |S^o| = |V(G) \cup (\bigcup_{v\in V(G)} S_v^o)| = |V(G)| + \sum_{v\in V(G)} |S_v^o| = |V(G)| + |V(G)||S_v^o|$$
$$= |V(G)|(1 + |S_v^o|) \geq |V(G)|\left(1 + \gamma_{sp}(H^v)\right), \forall v \in V(G).$$

Therefore, $\gamma_{spc}(G \circ H) = |V(G)|\left(1 + \gamma_{sp}(H)\right)$.□

The following lemma will be used in the characterization of super connected dominating set in the Cartesian product of two graphs.

Lemma 2.10*Let G and H be nontrivial connected graphs. If C' is a super connected dominating set of $G \square H$, then $C = C' \cup C''$ is a super connected dominating set of $G \square H$ for all $C'' \subset V(G \square H) \setminus C'$.*

Proof: Let $C = C' \cup C''$ where $C'' \subset V(G \square H) \setminus C'$. If $C'' = \emptyset$, then $C = C' \cup C''$ is a super connected dominating set of $G \square H$ is clear. If $C'' = \emptyset$, then let $(u, a) \in V(G \square H) \setminus C'$. Suppose that $(u, a) \in C''$. Since C' is a connected dominating set of $G \square H$, there exists $(v, a) \in C'$ such that $(u, a)(v, a) \in E(G \square H)$, that is, $C = C' \cup C''$ is connected. Suppose that $(u, a) \notin C''$. Since $(u, a) \notin C'$, $(u, a) \notin C$, and so $(u, a) \in V(G \square H) \setminus C$. Since C' is a super dominating set of $G \square H$, there exists $(v, a) \in C'$ such that $N_{G \square H}(v, a) \cup (V(G \square H) \setminus C') = \{(u, a)$. Since $C' \subset C$, it follows that for each $(u, a) \in V(G \square H) \setminus C$, there exists $(v, a) \in C$ such that $N_{G \square H}(v, a) \cup (V(G \quad H) \setminus C) = \{(u, a)\}$. This implies that C is a super dominating set of $G \square H$. Accordingly, C is a super connected dominating set of $G \square H$.□

The next result is the characterization of super connected dominating set in the cartesian product of two graphs.

Theorem 2.11*Let G and H be a nontrivial connected graphs. A nonempty subset C of $V(G\square H)$ is a super connected dominating set of $G\square H$ if and only ifS_G andS_H are super connected dominating sets of G and H respectively, and $S_G' \subset V(G) \setminus S_G$ and $S_{H'} \subset V(H) \setminus S_H$,one of the following statements is satisfied:*

(i) $C = S_G \times V(H)$
(ii) $C = V(G) \times S_H$
(iii) $C = [S_G \times V(H)] \cup [S_G' \times S]$ *where* $S \subset V(H)$.
(iv) $C = [V(G) \times S_H] \cup [S \times S_{H'}]$*where* $S \subset V(G)$.

Proof: Suppose that a nonempty subset C of $V(G\square H)$ is a super connected dominating set of $G\square H$. Let $(u, a) \in V(G\square H) \setminus C$. Since C is connected, there exists $(v, b) \in C$ such that $(v, b)(u, a) \in E(G\square H)$. This means that either $uv \in E(G)$ and $a = b$ or $v = u$ and $ab \in E(H)$.

Case 1: If $uv \in E(G)$ and $a = b$, then for each $u \in V(G) \setminus S_G$, there exists $v \in S_G \subset V(G)$ such that $vu \in E(G)$ and so, S_G is connected. Since C is super dominating set, for each $(u, a) \in V(GH) \setminus C$ there exists $(v, a) \in C$ such that $\$N_{\{G\square H\}}(v, a) \cap (V(G\square H) \setminus C) = \{(u, a)\}$. Thus, for each $u \in V(G) \setminus S_G$ there exists $v \in S_G$such that $\$N_G(v) \cap (V(G) \setminus S_G) = \{u\}$. This implies that S_G super dominating set of G. Since S_G is connected, S_G is a super connected dominating set of G. This clearly implies that $S_G \times V(H)$ is a super connected dominating set of $G\square H$. The proof of statement (i) is satisfied if we set $C = S_G \times V(H)$. Suppose *that* $C \neq S_G \times V(H)$. Since $S_G \times V(H)$ is a super connected dominating set of $G\square H$, let $C' = S_G \times V(H)$. In view of Lemma 2.11, $C = C' \cup C''$ is a super connected dominating set of $G\square H$ for all $C'' \subset V(G\square H) \setminus C'$. Let $S_G' \subset V(G) \setminus S_G$ and $S \subset V(H)$. Then $S_G' \times S \subset (V(G) \setminus S_G) \times V(H) = V(G\square H) \setminus \left(S_G \times V(H)\right) = V(G\square H) \setminus C'$. Thus, $S_G' \times S \subset V(G\square H) \setminus C'$. Set $C'' = S_G' \times S$. Then $C = C' \cup C'' = [S_G \times V(H)] \cup [S_G' \times S]$. This proves statement (iii).

Case 2: If $v = u$ and $ab \in E(H)$, then for each $a \in V(H) \setminus S_H$, there exists $b \in S_H \subset V(H)$ such that $ab \in E(H)$ and so, S_H is connected. Since C is
super dominating set, for each $(u, a) \in V(G\square H) \setminus C$ there exists $(v, a) \in C$ such that $N_{\{G\square H\}}(v, a) \cap (V(G\square H) \setminus C) = \{(u, a)\}$. Thus, for each $a \in V(H) \setminus S_H$ there exists $b \in S_H$ such that $N_H(b) \cap (V(H) \setminus S_H) = \{a\}$. This implies that S_H super dominating set of H. Since S_H is connected, S_H is a super connected dominating set of H. This clearly implies that $V(G) \times S_H$is a super connected dominating set of $G\square H$. The proof of statement (ii) is satisfied if we set $C = V(G) \times S_H$. Suppose that $C \neq (G) \times S_H$. Since $V(G) \times S_H$ is a super connected dominating set of $G\square H$, let $C' = V(G) \times S_H$. In view of Lemma 2.11, $C = C' \cup C''$ is a super connected dominating set of $G\square H$ for all $C'' \subset V(G\square H) \setminus C'$. Let $S_H' \subset V(H) \setminus S_H$and $S \subset V(G)$.

Then $S \times S_H' \subset V(G) \times (V(H) \setminus S_H) = V(G\square H) \setminus (V(G) \times S_H) = V(G\square H) \setminus C'$. Thus, $S \times S_H' \subset V(G\square H) \setminus C'. Set\ C'' = S \times S_H'$. Then $C = C' \cup C'' = [V(G) \times S_H] \cup [S \times S_H']$. This proves statement (iv).

For the converse, suppose that statement (i) is satisfied. Since S_G is a connected dominating sets of G, it follows that $S_G \times V(H)$ is a connected dominating set of $G\square H$. Thus, C is a connected dominating set of $G\square H$. Now, let $u \in V(G) \setminus S_G$. Since S_G is super dominating set, there exists $v \in S_G$ such that $N_G(v) \cap (V(G) \setminus S_G) = \{u\}$. Let $(u,a) \in V(G\square H) \setminus C$. Since $v \in S_G$, there exists $(v,a) \in C$ such that $N_{G\square H}(v,a) \cap (V(G\square H) \setminus C) = \{(u,a)\}$. This implies that C is a super dominating set of $G\square H$. Accordingly, C is a super connected dominating set of $G\square H$. Similarly, if statement (ii) is satisfied then C is a super connected dominating set of $G\square H$.

Suppose that statement (iii) is satisfied. Then $C = [S_G \times V(H)] \cup [S_G' \times S]$ where $S \subset V(H)$. Let \$$C' = S_G \times V(H)$. Then C' is super connected dominating set of $G\square H$ by (i). Let $C'' = S_G' \times S$. Then $C'' \subset V(G\square H) \setminus C'$. In view of Lemma 2.11, $C = C' \cup C''$ is a super connected dominating set of $G\square H$. Similarly, if statement (iv) is satisfied then C is a super connected dominating set of $G\square H$. □

The next result is a consequence of Theorem 2.11.

Corollary 2.12*Let G and H be nontrivial connected graphs. Then* $\gamma_{spc}(G\square H) = min\{\gamma_{spc}(G)|V(H)|, |V(G)|\gamma_{spc}(H)\}$.

Proof: Suppose that C is a super connected dominating set of $G\square H$. Then by Theorem 2.12(i) $C = S_G \times V(H)$ where S_G is a super connected dominating set of G or (ii) $C = V(G) \times S_H$ where S_H is a super connected dominating set of H or (iii) $C = [S_G \times V(H)] \cup [S_G' \times S]$ where $S \subset V(H) or\ (iv) C = [V(G) \times S_H] \cup [S \times S_H']$ where $S \subset V(G)$. We only need to consider statements (i) and (ii) since statements (iii) and (iv) are super sets of statements (i) and (ii) respectively. Consider the following cases:

Case 1: Suppose that $C = S_G \times V(H)$ where S_G is a super connected dominating set of G.Then$\gamma_{spc}(G\square H) \leq |C| = |S_G \times V(H)| = |S_G||V(H)|$ for all super connected dominating set S_G of G. Thus, $\gamma_{spc}(G\square H) \leq \gamma_{spc}(G)|V(H)|\ as\ inequality$ (1).

Case 2: Suppose that $C = V(G) \times S_H$ where S_H is a super connected dominating set of H.Then$\gamma_{spc}(G\square H) \leq |C| = |V(G) \times S_H| = |V(G)||S_H|$ for all super connected dominating set S_H of H. Thus, $\gamma_{spc}(G\square H) \leq |V(G)|\gamma_{spc}(H)$.
By Case 1 and Case 2, $\gamma_{spc}(G\square H) \leq min\{\gamma_{spc}(G)|V(H)|, |V(G)|\gamma_{spc}(H)\}$.

Suppose that C^o is the minimum super connected dominating set of $G\square H$. Consider the following cases.

Case 1: Suppose that $C^o = S_G \times V(H)$ where S_G is a super connected dominating set of G. Then $\gamma_{spc}(G\square H) = |C^o| = |S_G \times V(H)| = |S_G||V(H)| \geq \gamma_{spc}(G)|V(H)|. Thus, \gamma_{spc}(G\square H) \geq \gamma_{spc}(G)|V(H)|$.

Case 2: Suppose that $C^o = V(G) \times S_H$ where S_H is a super connected dominating set of H. Then \$$\gamma_{spc}(G\square H) = |C^o| = |V(G) \times S_H| = |V(G)||S_H| \geq |V(G)|\gamma_{spc}(H)$. Thus, $\gamma_{spc}(G\square H) \geq |V(G)|\gamma_{spc}(H)$.
By Case 1 and Case 2, $\gamma_{spc}(G\backslash square\ H) \geq min\{\gamma_{spc}(G)|V(H)|, |V(G)|\gamma_{spc}(H)\}$ as inequality (2).By combining inequality (1) and (2), we obtain the desired result. □

3CONCLUSION AND RECOMMENDATIONS

In this work, we introduced a new parameter of domination in graphs - the super connected domination in graphs. The super connected domination in the join, corona, and Cartesian product of two graphs were characterized. The exact super connected domination number resulting from these binary operations of two graphs were computed. This study will pave a way to new research such bounds and other binary operations of two graphs. Other parameters involving super connected domination in graphs may also be explored. Finally, the characterization of a super connected domination in graphs and its bounds is a promising extension of this study.

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