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ON βg^* -MAPS AND βg^* HOMEOMORPHISMS IN TOPOLOGICAL SPACES

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Abstract: This paper deals with βg^*-closed maps, βg^*-open maps, βg^*-homeomorphism, βg^*c-homeomorphism and study their properties. Using these new types of maps, several characterizations and properties have been obtained.
Keywords: βg^*-closed maps, βg^*-open maps, βg^*- homeomorphism, and βg^*c- homeomorphism.

1. INTRODUCTION

In 1970, Levine[6] introduced the concept of generalized closed set in topological spaces. In 2000, Veerakumar[9] introduced the concept of g* closed set and their continuity. Andrijevic[1] introduced semi pre open set(β open set) in general topology. Generalized closed mappings were introduced and studied by Malghan[7]. The definitions of some of them which are used our present study are given. The purpose of this paper is to introduce the concept of new class of maps called βg^*-closed maps and βg^*-open maps. Further βg^*-homeomorphism, βg^*c- homeomorphism are introduced and their properties are discussed.

2. PRELIMINARIES

Definition 2.1 A subset (X,τ) is said to be

1) g-closed [6] set if, $cl(A) \subseteq U$ whenever $A\subseteq U$ and U is open in X.
2) βg^*-closed [4] set if, $gcl(A) \subseteq U$ whenever $A\subseteq U$ and U is β-open in X

The complements of the above mentioned closed sets and their respective open sets.

Definition 2.2 A map f: X→Y is said to be

1) Continuous function if $f^{-1}(V)$ is closed in X for every closed set V in Y.

2) g-Continuous function [2], if $f^{-1}(V)$ is g-closed in X for every closed set V in Y.

3) Regular continuous [3] if $f^{-1}(V)$ is r closed subset in (X,τ) for every closed subset V in (Y,σ).

4) βg^*-Continuous function [5], if $f^{-1}(V)$ is βg^*-closed in X for every closed set V in Y

Definition 2.3 [8] A bijective function f: $(X,\tau)\to(Y,\sigma)$ is called

i) g-homeomorphism if both f and $f^1$are g-continuous.

3. βg^*-CLOSED MAPS

In this section the concepts of βg^*-closed maps are introduced and their basic properties are obtained.

Definition 3.1: A function f: (X, τ)→(Y, σ) is called βg^*- closed, if the image of each closed set in (X, τ) is a βg^*- closed set in (Y, σ).

Theorem 3.2: (i) Every closed map is βg^*- closed.

(ii) Every regular-closed map is βg^*- closed.

Proof: Let f: (X, τ)→(Y, σ) be a closed map. Let F be a closed subset of X. Since f is closed, f(F) is a closed set in Y. Then f(F) is a βg^*- closed set in Y. Hence f is βg^*- closed.

Proof of (ii) is similar to (i).

Remark 3.3: The converse of the above theorem need not be true, which is verified by the following examples.

Example 3.4: (i) Let X= Y= {a, b, c} with the topologies τ = {X, ϕ, {a}} and σ = {Y, ϕ, {a, b}} .Let the map f: (X, τ)→(Y, σ) be defined by f(a)=b, f(b)=a, f(c)=c. Then this function is βg^*- closed but not closed, as the image of the closed set {b, c} in X is {a, c} which is not a closed set in Y.

(ii) Let X= Y= {a, b, c} with the topologies τ = {X, ϕ, {a}} and σ = {Y, ϕ, {a}, {b, c}} .Let the map f: (X, τ)→(Y, σ) be defined by f(a)=b, f(b)=a, f(c)=c. Then this function is βg^*- closed but not regular-closed, as the image of the closed set {b, c} in X is {a, c} which is not a regular-closed set in Y.

Theorem 3.5: A function f: (X, τ)→(Y, σ) is βg^*- closed, if and only if for each subset S of (Y, σ) and for each open set U in (X, τ) containing f^{-1}(S), there exists a βg^*- open set V of (Y, σ) containing S such that S⊆V and f^{-1}(V)⊆U.

Proof: Necessity: Suppose that f is βg^*- closed, let S be a subset of (Y, σ) and U be an open set of (X, τ) such that f^{-1}(S) ⊆U. Now X−U is closed in (X, τ). Since f is βg^*- closed, f(X−U) is βg^*- closed in (Y, σ). Take V= Y−(f(X−U)). Then V is βg^*- open in (Y, σ). Since f^{-1}(S) ⊆U, S⊆f(U), S⊆V and f^{-1}(V)= f^{-1}(Y−(f(X−U))) ⊆U.

Sufficiency: Let F be closed subset of (X, τ).Then X−F is open in (X, τ) and f^{1}(Y−f(F)) ⊆X−F. By hypothesis, there exists a βg^*- open set V of (Y, σ), such that Y−f(F) ⊆V and f^{-1}(V) ⊆X−F and so F⊆X−f^{-1}(V). Then Y−V⊆f(F) ⊆f(X−f^{-1}(V)) ⊆Y−V. That is f(F) = Y−V. Hence f(F) is βg^*- closed in (Y, σ) and hence f is βg^*- closed.

Remark 3.6: The composition of two βg^*- closed maps need not be βg^*- closed map in general and this is shown by the following example.

Example 3.7: Let X= Y = Z= {a, b, c} ,τ={ X, ϕ,{a}, {a, b}}, σ = { Y, ϕ, {a, b}} and μ = { Z, ϕ, {b}, {c}, {b, c}}. Let f: (X, τ)→(Y, σ) and g: (Y, σ) → (Z, μ) be the identity maps. Then f and g are βg^*- closed maps, but their composition g∘f: (X, τ) → (Z, μ) is not βg^*- closed map, because F={b, c} is closed in (X, τ), but (g∘f)(F) = (g∘f)({b, c})= {b, c} which is not βg^*- closed in (Z, μ).

Theorem 3.8: If f: (X, τ)→(Y, σ) is closed map and g: (Y, σ)→(Z, μ) is βg^*- closed map. Then the composition g∘f: (X, τ) →(Z, μ) is βg^*- closed map.

Proof: Let F be any closed set in (X, τ). Since f is a closed map, f(F) is closed set in (Y, σ). Since g is βg^*- closed map, g(f(F)) = (g∘f)(F) is βg^*- closed set in (Z, μ). Thus g∘f is βg^*- closed map.

Theorem 3.9: Let f: (X, τ)→(Y, σ) and g: (Y, σ) → (Z, μ) be two functions such that g∘f: (X, τ) → (Z, μ) is βg^*- closed. Then if

(i) f is continuous and surjective, then g is βg^*- closed.

(ii) g is βg^*- irresolute and injective, then f is βg^*- closed.

(iii) f is βg^*- continuous, surjective and (X, τ) is a βg^*- space, then g is βg^*- closed.

Proof: (i) Let V be a closed set in (Y, σ). Then f^{-1}(V) is closed in X, since f is continuous. And (g∘f)(f^{-1}(V)) is βg^*- closed in (Z, μ), since g∘f is βg^*- closed. But (g∘f)(f^{-1}(V))= g(f(f^{-1}(V))) = g(V), since f is surjective. Hence g(V) is βg^*- closed in (Z, μ) and hence g is βg^*- closed.

(ii) Let U be a closed set in (X, τ). Since g∘f is βg^*- closed, (g∘f)(U) is βg^*- closed in (Z, μ). Since g is βg^*- irresolute, g^{-1}((g∘f)(U)) = g^{-1}(g(f(U))) is βg^*- closed in (Y, σ). Since g is injective, g^{-1}(g(f(U))) = f(U). Hence f(U) is βg^*- closed in (Y, σ) and hence f is βg^*- closed.

(iii) Let A be a closed set in (Y, σ). Since f is βg^*- continuous, f^{-1}(A) is βg^*- closed in (X, τ). As (X, τ) is a βg^*- space, f^{-1}(A) is closed in (X, τ) and by assumption (g∘f)(f^{-1}(A))= g(A) is βg^*- closed in (Z, μ). Therefore g is βg^*- closed.

Theorem 3.10: A function f: (X, τ)→(Y, σ) is βg^*- closed if and only if βg^*-cl(f(A))⊆f(cl(A)) for every subset A of X.

Proof: Suppose f: (X, τ)→(Y, σ) is a βg^*- closed map. Let A be a subset of X. Then cl(A) is a closed set in (X, τ). Since f is βg^*- closed f(cl(A)) is a βg^*- closed set containing f(A). Since βg^*- cl(f(A)) is the smallest βg^*- closed set containing f(A) we have βg^*- cl(f(A))⊆f(cl(A)).

Conversely, let F be a closed set in X. By our assumption βg^*- cl(f(F))⊆f(cl(F))=f(F). But f(F) ⊆ βg^*-cl(f(F)) implies βg^*- cl(F)=f(F).Therefore f(F) is βg^*- closed implies f is βg^*- closed map.

Definition 3.11: A function f: (X, τ)→(Y, σ) is called βg^*- open, if the image of each open set of (X, τ) is βg^*- open in (Y, σ).

Theorem 3.12: For any bijective map f: (X, τ)→(Y, σ), the following statements are equivalent:

(i) f^{-1}: (Y, σ) → (X, τ) is βg^*- continuous.

(ii) f is βg^*- closed.

(iii) f is βg^*- open.

Proof: (i) ⇒ (ii) Let U be a closed set in (X, τ). By assumption, $(f^{-1})^{-1}$(U)= f(U) is βg^*- closed in Y and so f is βg^*- closed.

(ii) ⇒(iii) Let F be an open set in (X, τ). Then X−F is closed in X. By assumption, f(X−F)= Y−f(F) is βg^*- closed in Y and hence f(F) is βg^*- open in Y. Hence f is βg^*- open.

(iii) ⇒ (i) Let F be an open set in (X, τ). By assumption, f(F) is βg^*- open in Y. But f(F)= $(f^{-1})^{-1}$(F). So $(f^{-1})^{-1}$(F) is βg^*- open in Y. Hence f^{-1} is βg^*- continuous.

Theorem 3.13: A function f: (X, τ)→(Y, σ) is βg^*-open if and only if f(int(A)) ⊆ βg^*- int(f(A)) for every subset A of X.

Proof: Suppose f: (X, τ)→(Y, σ) is βg^*-open. Now int(A) is open in (X, τ). Since f is βg^*-open, f(int(A)) is a βg^*-open set contained in f(A), Therefore f(int(A)) ⊆ βg^*-int(f(A)).

Conversely, assume that f(int(A)) ⊆ βg^*-int(f(A)). To prove that f is a βg^*-open map. Let U be open in (X, τ). Then int U=U. By assumption f(U) ⊆ βg^*-int(f(U)). But βg^*-int(f(U)) ⊆f(U) implies f(U)= βg^*-int(f(U)) which implies f(U) is βg^*- open in (Y, σ). Therefore f is βg^*-open map.

4. βg^*- HOMEOMORPHISM

In this section, a new class of maps namely, βg^*- homeomorphisms and βg^*c- homeomorphisms are introduced. Further it is proved that the set of all βg^*c-homeomorphisms forms a group under the operation composition of maps.

Definition 4.1: A function f: (X, τ)→(Y, σ) is called a βg^*- homeomorphism, if f is bijective and both f and f^{-1} are βg^*-continuous.

In other words A function f: (X, τ)→(Y, σ) is called a βg^*- homeomorphism, if f is bijective and both βg^*- open and βg^*-continuous.

Definition 4.2: A function f: (X, τ)→(Y, σ) is called a βg^*c- homeomorphism, if f is bijective and both f and f^{-1} are βg^*-irresolute.

Theorem 4.3: Suppose f is bijective, then the following conditions are equivalent:

(i) f is βg^*- closed and βg^*- continuous.

(ii) f is βg^*- open and βg^*- continuous.

(iii) f is a βg^*- homeomorphism.

Proof: (i) ⇒(ii) Now f is bijective, βg^*- closed and βg^*- continuous. Let A be an open set in X, Then X –A is closed in X and f(X –A) is βg^*- closed in Y. That is Y –f(A) is βg^*- closed in Y. Then f(A) is βg^*- open in Y. Hence f is βg^*- open.

(ii) ⇒(iii) Now f is a bijective, βg^*- open and βg^*-continuous map. Hence f^{-1} is also βg^*-continuous, by theorem 3.12. Thus f is a βg^*- homeomorphism.

(iii) ⇒(i) Let f be a βg^*- homeomorphism. Since f^{-1} is βg^*- continuous and bijective, f is βg^*-closed.

Theorem 4.4: Every homeomorphism is βg^*- homeomorphism but not conversely.

Proof: Let f: (X, τ)→(Y, σ) be a homeomorphism. Then f and f^{-1} are continuous and f is bijection. Since every continuous function is βg^*- continuous, f and f^{-1} are βg^*- continuous. Hence f is βg^*- homeomorphism.

Remark 4.5: The converse of the above theorem need not be true as seen from the following example.

Example 4.6: Let X= Y= {a, b, c} with topologies τ={ X, ϕ,{b}, {a, c}} and σ= { Y, ϕ, {a}}. The identity function f: (X, τ)→(Y, σ) is a βg^*- homeomorphism but not homeomorphism. Since A={a} is open in Y but f^{-1}(A)= {a} is not open in X. Thus f is not continuous. Hence f is not homeomorphism.

Theorem 4.7: Every regular homeomorphism is βg^*- homeomorphism but not conversely.

Proof: The proof follows from the theorem 3.2.

Example 4.8: Let X= Y= {a, b, c} with topologies τ={ X, ϕ,{b}, {a, c}} and σ= { Y, ϕ, {a}}. The identity function f: (X, τ)→(Y, σ) βg^*- homeomorphism but not regular homeomorphism. Since A={a} is open in Y but f^{-1}(A)={a} is not regular open in X. Thus f is not regular continuous. Hence f is nor regular homeomorphism.

Remark 4.9: The composition of two βg^*- homeomorphism need not be a βg^*- homeomorphism in general as seen from the following example.

Example 4.10: Let X= Y =Z={a, b, c} with topology τ={ X, ϕ, {b}}, σ= { Y, ϕ, {b}, {c}, {b, c}, {a, c}} and μ = { Z, ϕ, {a, b}}.Let f: (X, τ)→(Y, σ) and g: (Y, σ)→ (Z, μ) be identity maps. Then f and g are βg^*- continuous but g∘f:(X, τ)→(Z, μ) is not βg^*- continuous, since f^{-1}({c})={c} is not βg^*-open in (X, τ).

Theorem 4.11: If f: (X, τ)→(Y, σ) and g:(Y, σ) → (Z, μ) are βg^*c- homeomorphisms then g∘f: (X, τ)→ (Z, μ) is also a βg^*c- homeomorphism.

Proof: Let U be a βg^*- open set in (Z, μ). Now $(g∘f)^{-1}$(U)= $f^{-1}(g^{-1}(U))=f^{-1}(V)$ where V=g^{-1}(U). By hypothesis, V is a βg^*- open set in (Y, σ) and again by hypothesis, f^{-1}(V) is a βg^*- open set in (X, τ). Therefore g∘f is βg^*- irresolute. Also for a βg^*- open set G in (X, τ), we have (g∘f)(G)= g(f(G))= g(W), where W=f(G). By hypothesis, f(G) is βg^*-open in (Y, σ) and again by hypothesis, g(W) is βg^*- open in (Z, μ). Therefore $(g∘f)^{-1}$ is βg^*- irresolute. Hence g∘f is a βg^*c- homeomorphism.

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