Volume 6, No.2, FEBRUARY 2019
Journal of Global Research in Mathematical Archives
UGC Approved Journal
RESEARCH PAPER
Available online at http://www.jgrma.info
ISSN 2320 – 5822

ON PAIRS OF DISJOINT DOMINATING SETS IN THE COMPOSITION OF GRAPHS

Edward M. Kiunisala
Mathematics Department
College of Arts and Sciences
Cebu Normal University
Cebu City, Philippines

Abstract: In this paper, we investigate pairs of disjoint dominating sets *A* and *B* in the composition of graph, where *B* is either an independent or a total dominating set in the composition of graph.

Subject Classication: 05C69

Keywords: dominating set, independent dominating set, total dominating set, pair of disjoint dominating sets

1 INTRODUCTION

Throughout this study, we only consider graphs which are finite, simple andundirected. The symbols *V(G)* denotes the vertex set and *E(G)* denotes theedge set of *G*. The order of *G* refers to the cardinality of *V(G)* and the size of*G* refers to the cardinality of E(G). The symbol $|V(G)|$ denotes the order of*G* and $|E(G)|$ denotes the size of G. If $|E(G)| = 0$, then *G* is an *empty graph.*An empty graph of order n is denoted by K_n. If *V(G)* is singleton, *G* is calleda trivial graph.

The *composition*$G[H]$ of *G* and *H* is the graph with $V(G[H]) = V(G) \times V(H)$ and $(u; v)(u', v') \in E(G[H])$ if and only either $uu' \in E(G)$ or $u = u'$ and$vv' \in V(H)$. For any $v \in V(G)$, $G - v$ is the resulting graph after removingfrom *G* the vertex *v* and all edges of *G* incident to *v*.

Two distinct vertices *u* and *v* of *G* are neighbors in *G* if $uv \in E(G)$. Theclosed neighborhood $N_G[v]$ of a vertex *v* of *G* is the set consisting of *v* and everyneighbor of *v* in *G*. A *dominating* set in *G* is any $S \subseteq V(G)$for which $N_G[S] = V(G)$. The minimum cardinality of a dominating set is called the *dominationnumber* of *G*, denoted by $\gamma(G)$. Any dominating set in *G* of cardinality $\gamma(G)$ isreferred to as a γ-set in *G*. A dominating set *S* is an *independent dominatingset* if $uv \notin E(G)$for all $u,v \in S$. The minimum cardinality of an independentdominating set is called *the independence domination number* of *G*, denoted by$\gamma_i(G)$. Any independent dominating set in *G* of cardinality $\gamma_i(G)$ is referred toas a γ_i-set in G. A total dominating set S if for each $u \in S$ there is $v \in S$ such that$uv \in E(G)$. The minimum cardinality of a total dominating set is calledthe *total domination number* of *G*, denoted by $\gamma_t(G)$. Any total dominatingset in G of cardinality $\gamma_t(G)$. is referred to as a γ_t-set in *G*. The symbols $\mathcal{D}(G)$,$\mathcal{I}(G)$ and $\mathcal{T}(G)$ are used to denote the collection of all dominating sets, thecollection of all independent dominating sets, and the collection of all total dominating sets in G, respectively. The reader may refer to [1, 3, 5, 6, 7,8, 10, 11, 24, 25] for the fundamental concepts of domination theory, and to[3, 12, 25] for its applications.

Domination is one of the most well-studied concepts in graph theory (see[11]). The reader is referred to [1, 3, 5, 6, 7, 8, 10, 11, 24, 25] for the fundamental concepts and recent developments of the domination theory, and to [3, 12, 15,25] for its various applications.

In 1962, O. Ore gave the classical result which can be stated as follows: *Ifa graph G has no isolated vertices and S is a minimum dominating set, then*$V(G) \setminus S$ *is a dominating set in G.* It has motivated the introduction of theconceptof inverse domination (by V.R. Kulli and S.C. Sigarkanti [21]) as wellas theconcept of disjoint domination (by S.M. Hedetniemi, S.T. Hedetniemi,R.C. Laskar, L. Markus, and P.J. Slater [14]). A subset $S \subseteq V(G)$ is an *inversedominating set* in *G* if *S* is a dominating set in *G* and there is a minimumdominating set *D* in *G* such that $S \cap D = \emptyset$. The minimum cardinalityof an inverse dominating set in *G* is the *inverse domination number of G*,which is denoted by $\gamma'(G)$. A pair (*S*,*D*) of dominating sets in *G* is a dd-*pair*if $S \cap D = \emptyset$. We denote by $\mathcal{DD}(G)$ the collection of all dd-pairs in *G*.The minimum sum $|S| + |D|$ among all dd-pairs (*S*, *D*) in *G* is the disjointdomination number of *G*, which is denoted by $\gamma\gamma(G)$. That is,

$$\gamma\gamma(G) = \min\{|S| + |D|: (S, D) \in \mathcal{DD}(G)\}.$$

A dd-pair (S, D) with $|S| + |D| = \gamma\gamma(G)$ is called a $\gamma\gamma$-pair.

Inverse domination is studied further in [9, 19, 22]. Disjoint domination isalso further investigated in [13, 15, 16, 20, 23].

2 COMPOSITION OF GRAPHS

For any connected graphs G and H, if $S \subseteq V(G[H])$ is a γ-set in $G[H]$, thenS_G is a γ-set in G. Consequently, $\gamma(G) \leq \gamma(G[H])$.

Theorem 2.1[6] *Let G and H be connected graphs. Then*$C = U_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$,*where* $S \subseteq V(G)$ *and* $T_x \subseteq V(H)$*for every* $x \in S$, *is adominating set in G[H] if and only if either*

(i) *S is a total dominating set in G or*

(ii) *S is a dominating set in G and T_x is a dominating set in H for every*$x \in S\ N_G(S)$.

Theorem 2.2 [5] *Let G and H be connected graphs. A subset* $C = U_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$,
is an independent dominating set in G[H] if and only if S is an independent dominating set in G and $T_x$is an independent dominating set
inH for every $x \in S$.

Corollary 2.3 *Let G and H be connected graphs. A subset* $C = U_{x\in S}(\{x\} \times T_x) \subseteq V(G[H])$, *is a* γ_i*-set in* $G[H]$ *if and only if S is a* γ_i*-set in G and T_x isa* γ_i*-set in H for every* $x \in S$.

Proof : Suppose that $C = U_{x\in S}(\{x\} \times T_x)$ is a γ_i-set in $G[H]$. By Theorem 2.2,S is an independent dominating set in G and T_x is an independent dominatingset in H for each $x \in S$. Suppose that S^* is a γ_i-set in G, and let $D \subseteq V(H)$be a γ_i-set in H. Define $C^* = U_{x\in S}(\{x\} \times D)$. By Theorem 2.2, $C^*$is anindependent dominating set in $G[H]$. Since C is a γ_i-set in $G[H]$, $|C| \leq |C^*|$.On the other hand,$|C^*| = |S^*||D| \leq |S||D| \leq |C|$. Thus, $|C| = |C^*|$and,consequently, $|S| = |S^*|$and $|D| = |T_x|$for all $x \in S$.This means that S is aγ_i-set in G and T_x is γ_i-set in H for every $x \in S$.
Similar arguments will prove the converse.■

Lemma 2.4*Let G and H be connected nontrivial graphs such that V(H) isdominated in H by a vertex* $v \in V(H)$.*If* $A \subseteq V(G)$*is an inverse independentdominating set in G, then* $A \times \{v\}$*is an inverse independent dominating setin G[H].*

Proof : Let $B \subseteq V(G)$ be a γ_i-set in G and $A \subseteq V(G)\backslash B$a dominating setin G. By Corollary 2.3, $B \times \{v\}$ is a γ_i-set in $G[H]$. Also, by Theorem 2.1,$A \times \{v\}$ is a dominating set in $G[H]$. Since $(A \times \{v\}) \cap (B \times \{v\}) = \emptyset$), $A \times \{v\}$is an independent inverse dominating set in $G[H]$. ■

Corollary 2.5*Let G and H be connected nontrivial graphs with* $\gamma(H) = 1$.
Then

$$\gamma(G) \leq \gamma_i'(G[H]) \leq \gamma_i'(G). \quad (1)$$

Define $S^\circ = S\backslash N_G(S)$ for any $S \subseteq V(G)$

Theorem 2.6*Let G and H be nontrivial connected graphs with* $\gamma(H) = 1$.

(i) *If H has (at least) two distinct vertices each of which dominates V(H),*
then$\gamma_i'(G[H]) = \gamma(G)$.

(ii) *If H has a unique vertex that dominates V(H), then*

$$\gamma_i'(G[H]) = \min\{|A| + |A^\circ \cap B|(\gamma'(H) - 1): A \in \mathcal{D}(G), B \in \mathcal{I}(G)$$

$$with\ \gamma_i(G) = |B|\}.$$

Proof : (*i*) Suppose that *H* has two distinct vertices *u* and *v* such that $N_H[u] = V(H) = N_H[v]$, and let $A, B \subseteq V(G)$ be γ-set and a γ_i-set, respectively, in*G*. Then $S = A \times \{u\}$and $D = B \times \{v\}$are a γ-set and a γ_i-set, respectively, in *G*[*H*]. Since $S \cap D = \emptyset$, S is a γ_i'-set in *G*[*H*]. Hence, $\gamma_i'(G[H] \leq |S| = |A| = \gamma(G)$.The desired equality follows from Inequality 1.

Suppose that H has a unique vertex *v* that dominates *V*(*H*), and let

$$\alpha = \min\{|A| + |A^\circ \cap B|(\gamma'(H) - 1): A \in \mathcal{D}(G), B \in \mathcal{I}(G)\ with\ \gamma_i(G) = |B|\}.$$

Let *A* and *B* be a dominating set and γ_i-set, respectively, in *G*, and let $v \in V(H)$ such that $N_H[v] = V(H)$. Choose $w \in V(H) \backslash \{v\}$ and a γ'-set $C \subseteq V(H)$in *H*. Since *H* has a unique dominating set, namely $\{v\}, v \notin C$.

Define $D = B \times \{v\}$and

$$S = \left(\cup_{u \in A \backslash B} \{(u, v)\}\right) \cup (\cup_{u \in (A \backslash A^\circ) \cap B} \{(u, w)\} \cup (\cup_{u \in A^\circ \cap B} (\{u\} \times C).$$

By Corollary 2.3, *D* is a γ_i-set in *G*[*H*]. Let $u \in A^\circ$. Then $T_u = \{x \in V(H): (u, v) \in S\}$is either *C* or $\{v\}$. In any case, T_u is a dominating set in *H*. ByTheorem 2.1,*S* is a dominating set in *G*[*H*]. Since $S \cap D = \emptyset$,*S*is an inverseindependent dominating set in *G*[*H*]. Thus,

$$\gamma_i'(G[H]) \leq |S| = |A| + |A^\circ \cap B|(\gamma'(H) - 1).$$

Since A and B are arbitrary, $\gamma_i'(G[H]) \leq \alpha$.

Let (*S*,*D*) be a di-pair in *G*[*H*] such that $|D| = \gamma_i(G[H])$ and $|S| = \gamma_i'(G[H])$.By Theorem 2.1, $S = U_{u \in A}(\{u\} \times T_u)$ and $D = U_{u \in B}(\{u\} \times T_u)$forsome dominating sets *A* and *B* in *G*. More particularly, by Corollary 2.3, *B*is aγ_i-set in *G*. Since $\gamma(H) = 1$, Corollary 2.3 implies that$|T_u| = 1$ for all$u \in B$ and $|D| = |B| = \gamma_i(G)$.Since S is a γ_i'-set, $|T_u| = 1$ for all $u \in A \backslash B$,in which case, we may assume that $T_u = \{v\} \subseteq V(H)$where $N_H[v] = V(H)$.Since $S \cap D = \emptyset$, for all $u \in A^\circ \cap B$, if $(u, w) \in D$, then $(u, w) \notin S$.Moreover,in view of Theorem 2.1(*ii*), for each such *u*, $T_u = \{x \in V(H): (u, x) \in S\}$ is a γ-set in *H*. Thus,

$$\begin{aligned} |S| &= \left|\cup_{u \in A \backslash B} (\{u\} \times T_u)\right| + \left|\cup_{u \in (A \backslash A^\circ) \cap B} (\{u\} \times T_u)\right| + |U_{u \in A^\circ \cap B}(\{u\} \times T_u)| \\ &\geq |A \backslash (A^\circ \cap B)| + |A^\circ \cap B|\gamma'(H) \\ &= |A| + |A^\circ \cap B|(\gamma'(H) - 1) \end{aligned}$$

So that $\gamma'(G[H] \geq \alpha$. ■

.

Corollary 2.7*Let G and H be connected nontrivial graphs. Suppose that H has a unique vertex that dominates*V(*H*). *Then*

(*i*) $\gamma_i'(G[H]) = \gamma_i'(G)$if and only if G has a γ_i'-set A_0 such that

$$|A_0| \leq |A \cap B|(\gamma'(H) - 1) + |A|$$

for all dominating sets A and γ_i-sets B in G

(ii) If$\gamma_i(G) = \gamma(G)$,then$\gamma_i'(G[H]) = \gamma(G)$.

Proof : (*i*) Let $A_0 \subseteq V(G)$be a γ_i'-set in G such that $|A_0| \leq |A \cap B|(\gamma'(H) - 1) + |A|$ for all dominating sets *A* and γ_i-sets B in G. By Theorem 2.6 andInequality 1 in Corollary 2.5, $\gamma_i'(G) = |A_0| \leq \gamma_i'(G[H]) \leq \gamma_i'(G)$. The converseis clear.

(ii) Let $B \subseteq V(G)$be a γ_i-set in *G*. Put *A* = *B*. Since $A^\circ = \emptyset$and $\gamma(G) = |A| = |A| + |A^\circ \cap B|(\gamma'(H) - 1)$,Theorem 2.6 implies $\gamma_i'(G[H]) \leq |A| = \gamma(G)$. The desired equality follows from Inequality 1 in Corollary 2.5. ■

Theorem 2.8*Let G and H be connected nontrivial graphs with* $\gamma(H) = 1$.
Then
$2\gamma(G) \leq \gamma\gamma_i(G[H]) \leq \gamma\gamma_i(G)$.(2)

More precisely,

(*i*) *if H has (at least) two distinct vertices each of which dominates V*(*H*),
then $\gamma\gamma_i(G[H]) = 2\gamma(G)$; *and*

(*ii*) *if Hhas a unique vertex that dominates V*(*H*), *then*

$$\gamma\gamma_i(G[H]) = \min\{|A| + |B| + |A^{\circ} \cap B|(\gamma'(H) - 1): A \in \mathcal{D}(G), B \in \mathcal{I}(G)\}.$$

Proof : There exists $v \in V(H)$ such that $N_H[v] = V(H)$. Let (*A*, *B*) be a $\gamma\gamma_i$-pair in *G*. Then $(A \times \{v\}, B \times \{v\})$is a *di*-pair in *G*[*H*]. Thus, $\gamma\gamma_i(G[H]) \leq |A \times \{v\}| + |B \times \{v\}| = |A| + |B| = \gamma\gamma_i(G)$.

If *H* has two distinct vertices that both dominate *V*(*H*), then Theorem 2.6(*i*) implies

$$2\gamma(G) \leq \gamma\gamma_i(G[H]) \leq \gamma(G[H]) + \gamma_i'(G[H]) = 2\gamma(G).$$

Suppose that *H* has a unique vertex *v* that dominates *V*(*H*). Let

$$\alpha = \min\{|A| + |B| + |A^{\circ} \cap B|(\gamma'(H) - 1): A \in \mathcal{D}(G), B \in \mathcal{I}(G)\}.$$

Let $w \in V(H)\backslash\{v\}$, (X, Y)a *di*-pair in *H*, and let *A* and *B* be a dominatingset and an independent dominating set, respectively, in *G*. Define

$$S = \left(\cup_{u \in A\backslash B} \{(u, v)\}\right) \cup \left(\cup_{u \in (A\backslash A^{\circ})\cap B} \{(u, w)\}\right) \cup \left(U_{u \in A^{\circ}\cap B}(\{u\} \times X)\right),$$

and $D = \left(U_{u \in A^{\circ}\cap B}(\{u\} \times Y)\right) \cup \left(\cup_{u \in B\backslash(A^{\circ}\cap B)} \{(u, v)\}\right)$.By Theorem 2.1, *S* is a dominating sets in *G*[*H*]. By Theorem 2.2, *D* is an independent dominating set in *G*. Moreover, $S \cap D = \emptyset$. Thus,

$$\gamma\gamma_i(G[H]) \leq |S| + |D|$$

$$= |A| + |B| + |A^{\circ} \cap B|(|X| + |Y| - 2).$$

Since *X* and *Y*are arbitrary$\gamma\gamma_i(G[H]) \leq |A| + |B| + |A^{\circ} \cap B|(\gamma\gamma_i(H) - 2)$.
Write $H = K_1 + H^*$_, where $\gamma(H^*) \geq 2$.So, $(\gamma\gamma_i(H) - 2) = \gamma'(H) - 1$.
Thus,

$$\gamma\gamma_i(G[H]) \leq |A| + |B| + |A^{\circ} \cap B|(\gamma'(H) - 1)$$

Since A and B are arbitrary, $\gamma\gamma_i(G[H]) \leq \alpha$.

To prove the converse, let (*S*,*D*) be a $\gamma\gamma_i$-pair in *G*[*H*]. There exists adominating set *A* in *G* and an independent dominating set *B* in *G* such that $S = \cup_{u \in A} (\{u\} \times T_u)$ and $D = \cup_{u \in B} (\{u\} \times T_u)$.If *A* is not a total dominating set in *G*, then T_u is a dominating set in *H*. Also, T_u is an independent dominating set in *H* for all $u \in B$. In particular, for each $u \in A^{\circ} \cap B, X = \{y \in V(H): (u, y) \in S\}$and $Y = \{y \in V(H): (u, y) \in D\}$constitute a *di*-pair in *H*. Since(*S*, *D*) is a$\gamma\gamma_i$-pair in *G*[*H*], we have for each $u \in A^{\circ} \cap B, |X| + |Y| \geq \gamma\gamma_i(H)$.For each$u \in (A\backslash A^{\circ}) \cap B, \{y \in V(H): (u, y) \in D\} = \{v\}$, and for each $u \in B\backslash A, \{y \in V(H): (u, y) \in D\} = \{v\}$. Thus,

$$\gamma\gamma_i(G[H]) = |S| + |D| \geq |A| + |B| + |A^{\circ} \cap B|(\gamma'(H) - 2) \geq \alpha.$$

This proves Statement (*ii*).■

Corollary 2.9*Let G and H be connected nontrivial graphs. Suppose that H has a unique vertex that dominates V*(*H*). *Then* (*i*) $\gamma\gamma_i(G[H]) = \gamma\gamma_i(G)$ *if and only if G has a* $\gamma\gamma_i$*-pair* (A_0, B_0) *such that* $|A_0| + |B_0| \leq |A| + |B| + |A^\circ \cap B|(\gamma'(H) - 1)$ *for all dominating sets A in G and independent dominating sets B in G.*

(*ii*) If $\gamma_i(G) = \gamma(G)$, *then* $\gamma\gamma_i(G[H]) = 2\gamma(G)$.

Example 2.10 *Let G be any connected nontrivial graph. Then*

(*i*) $\gamma_i'(G[K_{1,n}]) = \min\{|A| + (n-1)|A^\circ \cap B| : A \in \mathcal{D}(G), B \in \mathcal{I}(G)\}, |B| = \gamma_i(G)\}$ and
$\gamma\gamma_i(G[K_{1,n}]) = \min\{|A| + |B| + (n-1)|A^\circ \cap B| : A \in \mathcal{D}(G), B \in \mathcal{I}(G)\}$ for $n \geq 2$;

(*ii*) $\gamma_i'(G[K_p]) = \gamma(G)$ and $\gamma\gamma_i(G[K_p]) = 2\gamma(G)$ for $p \geq 2$.

Proposition 2.11 *For noncomplete connected graphs G and* $p \geq 2$,

$$\gamma_i'(K_p[G]) = \begin{cases} 1, & if\ \gamma(G) = 1, \\ 2, & otherwise. \end{cases}$$

Proof : If $\gamma(G) = 1$, then $\gamma_i'(K_p[G]) = \gamma(K_p) = 1$, by Theorem 2.6(*i*) and Corollary 2.9(ii). Suppose that $\gamma(G) \geq 2$. Then $\gamma(K_p[G]) \geq 2$ and, hence, $\gamma_i'(K_p[G]) \geq 2$. Let $D \subseteq V(K_p[G])$ be a γ_i-set in $K_p[G]$. By Corollary 2.3, $D = \cup_{v \in A} \{(u, v)\}$ for some γ_i-set *A* in *G* and $u \in V(K_p)$. Since G is nontrivial, $V(G) \backslash A \neq \emptyset$. Let $y \in V(G) \backslash A$ and put $S = \{(u, y), (x, y)\}$, where *u* and *x* are distinct vertices of K_p. Since *S* is a dominating set in $K_p[G]$ and $S \cap D = \emptyset$, *S* is an inverse independent dominating set in $K_p[G]$. Thus $\gamma'_i(K_p[G]) \leq 2$.).■

Corollary 2.12 *For noncomplete connected graphs G and* $p \geq 2$,

$$\gamma\gamma_i(K_p[G]) = \begin{cases} 2, & if\ \ \gamma(G) = 1, \\ 4, & otherwise \end{cases}.$$

Proof : If $\gamma(G) = 1$, then Inequality 2 yields $\gamma\gamma_i(K_p[G]) = 2$. Suppose that $\gamma(G) \geq 2$. Then $3 \leq \gamma\gamma_i(K_p[G]) \leq 4$.
If $\gamma\gamma_i(K_p[G]) = 3$, then $\gamma(G) = 1$, a contradiction. Thus, $\gamma\gamma_i(K_p[G]) = 4$. ■

Theorem 2.13 [6] *Let G and H be connected graphs. Then* $C = \cup_{x \in S} (\{x\} \times T_x) \subseteq V(G[H])$, *where* $S \subseteq V(G)$ and $T_x \subseteq V(H)$ *for each* $x \in S$, *is a total dominating set in* G[H] *if and only if*

(*i*) *S is a total dominating set in G or*

(*ii*) *S is a dominating set in G and* T_x *is a total dominating set in H for every* $x \in S \backslash N_G(S)$.

Theorem 2.14 *Let G and H be connected nontrivial graphs and* $\gamma(H) = 1$. *Then*

(*i*) if *H* has (at least) two distinct vertices each of which dominates *V*(*H*),
then $\gamma\gamma_t(G[H]) = \gamma(G) + \gamma_t(G)$;

(*ii*) *if H has a unique vertex that dominates V(H), then*

$$\gamma\gamma_t(G[H]) = \min\{|A| + |B| : A \in \mathcal{D}(G), B \in \mathcal{T}(G)\}.$$

Proof : (*i*) Suppose that *H* has distinct vertices *u* and *v* such that $N_H[u] = V(H) = N_H[v]$. Let $S, S^* \subseteq V(G)$ be a γ-set and a γ_t-set in *G*. Define $D = S \times \{u\}$ and $T = S^* \times \{v\}$. Then (*S*,*T*) is a *dt*-pair in *G*[*H*]. Thus,

$$\begin{aligned} \gamma(G) + \gamma_t(G) &= \gamma(G[H]) + \gamma_t(G[H]) \\ &\leq \gamma\gamma_t(G[H]) \\ &\leq |D| + |T| \\ &= \gamma(G) + \gamma_t(G), \end{aligned}$$

and $\gamma(G) + \gamma_t(G) = \gamma(G) + \gamma_t(G)$.

(*ii*) Define $\alpha = \min\{|A| + |B|: A \in \mathcal{D}(G), B \in \mathcal{T}(G)\}$. Let $A \in \mathcal{D}(G)$and $B \in \mathcal{T}(G)$.Let $v \in V(H)$be such that $N_H[v] = V(H)$ and $w \in V(H)\backslash\{v\}$.Then (*D*; *T*), where $D = A \times \{v\}$and $T = B \times \{w\}$, is a *dt*-pair in *G*[*H*]. Thismeansthat $\gamma\gamma_t(G[H]) \leq |D| + |T| = |A| + |B|$. Since *A* and *B* are arbitrary,$\gamma\gamma_t(G[H]) \leq \alpha$.

Let (*D*; *T*) be a $\gamma\gamma_t$-pair in *G*[*H*]. Then $D = \cup_{u\in A}(\{u\} \times T_u)$and $T = \cup_{u\in B}(\{u\} \times T_u)$for some dominating sets *A* and *B* in *G* and $T_u \subseteq V(H)$.Moreover, by Theorem 2.1, if *A* is not a total dominating set in *G*, then $T_u$is a dominating set for all $u \in A^\circ$. Also, by Theorem 2.13, if *B* is not a total dominating set in *G*, then T_u is a total dominating set in *H* for all $u \in B^\circ$.If
B is a total dominating set in *G*, then

$$|D| + |T| \geq |A| + |B| \geq \alpha.$$

Suppose that *B* is not a total dominating set in *G*. For each $u \in B^\circ$, if $T_u = \{y \in V(H): (u, y) \in T\}$, then $|T_u| \geq 2$so that $|\cup_{u\in B^\circ}(\{u\} \times T_u| \geq 2|B^\circ|$.For each $u \in B^\circ$, choose $v(u) \in V(G)$such that $uv(u) \in E(G)$.Define $C = (B\backslash B^\circ) \cup B^\circ \cup \{v(u): u \in B^\circ\}$.Then *C* is a total dominating set in *G*and

$$|D| + |T| \geq |A| + |C| \geq \alpha.$$

This completely proves the Statement (*ii*) of the theorem. ∎

3. REFERENCES

[1] R.B. Allan and R. Laskar, *On domination and independent domination numbers of a graph*, Discrete Mathematics, Vol.23, No.2, 73-76, 1978.

[2] V. Anusuya, R. Kala,*A Note on Disjoint Dominating Sets in Graphs*, Int.J. Contemp. Math. Sciences, Vol. 7, 2012.

[3] C. Berge, *Theory of Graphs and its Applications*, Methuen, London, 1962.

[4] F. Buckley, F. Harary. *Distance in Graphs*. Redwood City. CA: AddisonWesley. 1990.

[5] S.R. Canoy Jr. and M.A. Labendia, *Independent domination in graphs under some binary operations, to appear*.

[6] S.R. Canoy Jr. and C.E. Go, *Domination in the composition and Cartesian product of graphs*, AKCE, to appear.

[7] E. J. Cockayne, R. M. Dawes, and S. T. Hedetnieme. *Total domination in graphs.* Networks, 10(3):211-219, 1980.

[8] E. J. Cockayne and S. T. Hedetniemi, *Towards a Theory of Dominationin Graph*. Networks 7 (1977) 247-261.

[9] G.S. Domke, J.E. Dunbar, and L.R. Markus, *The inverse domination number of a graph*, ArsCombin., 72 (2004), 149-160.

[10] W. Duckworth and N.C. Wormald(2006*), On the independent domination number of random regular graphs*, Combinatorics, Probability and Computing, Vol.15, No.4, pp 513-522.

[11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in Graphs*. Marcel Dekker, Inc. New York(1998).

[12] T. Haynes, S.Hedetniemi and M. Henning, *Domination in graphs applied to electrical power networks*, J. Discrete Math. 15(4), 519-529, 2000.

[13] S.M. Hedetniemi, S.T. Hedetniemi, R.C. Laskar, L. Markus, and P.J. Slater, *Disjoint dominating sets in graphs*, manuscript 2007.

[14] Hedetniemi T.W., Hedetniemi S.T, RenuC.Laskar, Slater P.J., *Disjoint dominating sets*, Proceedings of Int. conf. on Disc. Math.(ICDM 2006) 88-102.

[15] M.A. Henning, C. Lowenstein, and D. Rautenbach, *Remarks about disjoint dominating sets*, Discrete Mathematics, 309 (2009), no. 23-24, 6451-6458.http://dx.doi.org/10.1016/j.disc.2009.06.017.

[16] M.A. Henning and J. Southey, *A note on graphs with disjoint dominatingand total dominating sets*, ArsCombin 89 (2008), 159-162.

[17] M.A. Henning and J. Southey, *A characterization of graphs with dis- joint dominating and total dominating sets*, QuaestionesMathematicae 32 (2009), 119-129.

[18] M.A. Henning, C. Lowenstein and D. Rautenbach, *Partitioning a graphinto a dominating set, a total dominating set, and something else*, Discus-sionesMathematicae: Graph Theory;2010, Vol. 30 Issue 4, p563.

[19] E.M. Kiunisala and F.P. Jamil, *Inverse domination numbers and disjoint domination numbers of graphs under some binary operations*, Applied Mathematical Sciences, Vol. 8, 2014, no. 107, 5303 - 5315.

[20] E.M. Kiunisala and F.P. Jamil, *On Pairs of Disjoint Dominating Sets in a graph*, International Journal of Mathematical Analysis, Vol. 10, 2016,no. 13, 623-637.

[21] V.R. Kulli and S.C. Sigarkanti, *Inverse domination in graphs*, Nat. Acad. Sci. Letters, 14 (1991) 473-475.

[22] KulliV.R.,andIyer R.R., *Inverse total domination in graphs*, J.Discrete Mathematical Sciences and Cryptography, Vol. 10 Number 5 (2007), pp 613-620.

[23] C.Lowenstein and D.Rautenbach, *Pairs of disjoint Dominating sets and the minimum degree of graphs*, Graphs and Combinatorics, 26 (2010),407-424. http://dx.doi.org/10.1007/s00373-010-0918-9

[24] O. Ore, *Theory of graphs*, Amer. math. Soc. Transl., 38(Amer. Math. Soc.,Providence, RI, 1962), 206-212.

[25] H. B. Walikar, B. D. Acharya and E. Sampathkumar, *Recent Develop-ments in the Theory of Domination in Graphs and its Applications*. MRI Lecture Notes in Mathematics 1 (1979).

