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INVERSE CLOSED DOMINATION ON THE UNITARY CAYLEY GRAPHS

Edward M. Kiunisala and Cristopher John S. Rosero

Mathematics Department
College of Arts and Sciences
Cebu Normal University, 6000
Cebu City, Philippines

Abstract: Let $(\Gamma,*)$ be a finite group and *e* be its identity. Let *S*be a generating set of *G* such that $e \neq S$and$a^{-1} \in S$for all $a \in S$. Then the Cayley Graph is defined by$G = (V, E)$, where $V = \Gamma$ and $E = \{(x, x * a) | x \in V, a \in S\}$, denoted by $Cay(\Gamma, S)$. The *Unitary Cayley Graph*, $X_n = Cay(\mathbb{Z}_n, U_n)$is defined by the additive group of the ring $\mathbb{Z}_n$of integers modulo *n* and the multiplicative group of $U_n$of its units.If we represent the elements of$\mathbb{Z}_n$by the integers$0,1,2, \ldots, n-1$,then it is known that$U_n = \{a \in \mathbb{Z}_n | \gcd(a, n) = 1\}$. So $X_n$has a vertexset $V(X_n) = \mathbb{Z}_n = \{0,1,2, \ldots, n-1\}$and the edge set $E(X_n) = \{\{a, a+b\} | a \in \mathbb{Z}_n \text{ and } b \in U_n\}$.

In this paper, the domination in graph is extended to a Unitary Cayleygraphs, in particular the inverse closed domination on the Unitary CayleyGraphs.

Mathematics Subject Classification: 05C69

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1. INTRODUCTION

Domination as a graph theoretic concept was first introduced by C. Bergein 1958 and O. Ore in 1962. It was O. Ore [12] who introduced the termdominating set and domination number. In 1977, E.J. Cockayne and S.T.Hedetniemi [5] presented a survey on published works in domination. Since apublication of the said survey, domination theory has been studied extensively.In their book, T.W. Haynes, S.T. Hedetniemi and P.J. Slater listed in [8] over1200 references in this topic including over 75 variations. The paper of Kulliand Sigarkanti [11] in 1991 which initiated the study of inverse dominationin graphs and further read in [7, 10, 14]. In this study we introduced a newdomination parameter, the inverse closed domination in graphs and give someimportant results.

A dominating set is called a *closed dominating set* if given a graph G, choose$v_1 \in V(G)$ and put $S_1 = \{v_1\}$. If $N_G[S_1] \neq V(G)$,choose $v_2 \in V(G) \backslash S_1$andput $S_2 = \{v_1, v_2\}$. Where possible, $k \geq 3$, choose $v_k \in V(G) \backslash N_G[S_{k-1}]$and put$S_k = \{v_1, v_{2,\ldots,} v_k\}$. There exists a positive k such that $N_G[S_k] = V(G)$. Thesmallest cardinality of a closed dominating set is called the *closed dominationnumber* of G, and denoted by $\bar{\gamma}(G)$.A close dominating set of cardinality $\bar{\gamma}(G)$is called $\bar{\gamma}$-*set* of G. A closed dominating set S is said to be in its canonicalform if it is written as $S = \{v_1, v_{2,\ldots,} v_k\}$where the vertices $v_k$satisfy theproperties given above.

Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \backslash D$is called an *inverse dominating set* with respect to D. The minimum cardinalityof inverse dominating set is called an *inverse domination number* of G and isdenoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is calledγ^{-1}-*set* of G. Motivated by the definition of inverse domination in graph, wedefine a new domination parameter. Let C be a minimum closed dominatingset in G. The closed dominating set$S \subseteq V(G) \backslash C$is called an *inverse closeddominating set* with respect to C. The minimum cardinality of an inverseclosed dominating set is called an *inverse closed domination number* of G andis denoted by $\bar{\gamma}^{-1}$. An inverse closed dominating set of cardinality$\bar{\gamma}^{-1}(G)$ is called $\bar{\gamma}^{-1}$-*set* of G.

In this paper, the domination in graph is extended to a Unitary Cayleygraphs, in particular the inverse closed domination of the Unitary Cayley Graphs.Let $(\Gamma,*)$ be a finite group and e be its identity. Let Sbe a generating set of Γ such that $e \notin S$and $a^{-1} \notin S$for all $a \in S$. Then the Cayley Graph is definedby $G = (V, E)$,where $V = \Gamma$and $E = \{(x, x * a) | x \in V, a \in S\}$ denoted by$Cay\ (\Gamma, S)$. The *Unitary Cayley Graph* $X_n = Cay\ (Z_n, U_n)$ is defined by theadditive group of the ring $\mathbb{Z}_n$, of integers modulo n and the multiplicativegroup of $U_n$of its units. If we represent the elements of$\mathbb{Z}_n$by the integers$0,1,2, \ldots, n-1$,then it is known that

$U_n = \{a \in \mathbb{Z}_n \mid \gcd(a,n) = 1\}$.So X_n has a vertex set $V(X_n) = \mathbb{Z}_n = \{0,1,2,\dots,n-1\}$and the edge set $E(X_n) = \{\{a, a+b\} \mid a \in \mathbb{Z}_n \text{ and } b \in U_n\}$. TamizhChelvam and Rani [15] obtained theconnected domination numbers for certain Cayley Graphs constructed on $\mathbb{Z}_n$for some generatingset of Z_n.

In this study, we attempt to find the inverse closed dominating sets in theUnitary Cayley Graphs.

2. PRELIMINARIES

Remark 2.1 *Tacbobo provided the following in [13].*

(*i*) *For a complete graph $K_n$with n vertices,$\bar{\gamma}(K_n) = 1$;*

(*ii*) *For a cycle $C_n$with n vertices, $n \geq 3, \bar{\gamma}(C_n) = \left\lceil \frac{n}{3} \right\rceil$;*

(*iii*) *For a complete bipartite graph $K_{m.n}$, $\bar{\gamma}(K_{m,n}) = 2$.*

Let $X_n = Cay(\mathbb{Z}_n, U_n)$be a unitary Cayley graph.

Theorem 2.2 [6] X_n is a bipartite if *n* is an even number.

Theorem 2.3[16] X_n is $\phi(n)$-regular for all *n*, where $\phi(n)$ denotes the Eulerfunction.

Theorem 2.4 [1] $X_n$is Eulerian for all $n \geq 3$.

The Euler's phi function $\phi(n)$, is defined to be the number of positive integers less than *n* that are relatively prime to *n* introduced by Leonhard Euler (1703-1783).

The theorem below is a formula for obtaining$\phi(n)$, based on the factorization of *n* as a product of primes.

Theorem 2.5 [2]
1.If *p* is prime number, then $\phi(n) = p - 1$.

2. If the integer *n* >1 has the prime factorization $n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_r^{k_r}$,
then$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdot \dots \left(1 - \frac{1}{p_r}\right) \cdot$

3.Let *p* be a prime and *a* be a positive integer. Then $\phi(p^a) = p^a - p^{a-1}$.

Theorem 2.6 [2] *The function $\phi$is a multiplicative function.*

Theorem 2.7 [4] *The following hold:*

1.For any integer $n = 2,\ \gamma(K_n) = \gamma^{-1}(K_n) = 1$.

2.For integers $m, n \geq 2,\ \gamma(K_{m,n}) = \gamma^{-1}(K_{m,n}) = 2$.

Theorem 2.8 [4] For an integer $n \geq 3, \gamma(C_n) = \gamma^{-1}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

3. CLOSED DOMINATION IN GRAPHS

Theorem 3.1 *If n is prime, then$\bar{\gamma}(X_n) = 1$.*

Proof : Suppose *n* is prime. Let *v* be a vertex of the unitary Cayley graph X_n. Since $|U_n| = \phi(n)$, let $n = \{1, v_2, \dots, v_{\phi(n)-1}, v_{\phi(n)}\}$. By the definition of unitary Cayley graph, $v \in V(X_n)$ is adjacent to a vertex $u \in V(X_n)$ if and only if $(u - v, n) = 1$.Since*n* is prime, this implies that $u - v$is relatively prime to *n*. Hence $u - v$is a unit in the ring $\mathbb{Z}_n$, and *u* and *v* are adjacent. Since *u*and *v* are arbitrary, each vertex of $X_n$is adjacent to every other vertex, and$X_n \cong K_n$.Hence, by Remark 2.1 (*i*), then $\bar{\gamma}(X_n) = 1$.

Theorem 3.2 *Let n be an integer such that $\phi(n) = 2$. Then $\bar{\gamma}(X_n) = \left\lceil \frac{n}{3} \right\rceil$.*

Proof : Suppose $\phi(n) = 2$. Then $|U_n| = 2$.Hence, $X_n$is 2-regular by Theorem2.3. Since X_n is connected for all n due to Theorem 2.4, we must have $X_n \cong C_n$,a cycle with n vertices. Therefore, by Remark 2.1 (ii), $\overline{\gamma}(X_n) = \left\lceil \frac{n}{3} \right\rceil$. ∎

Theorem 3.3 *If n is an integer such that* $n = 2^r, r \geq 2$, *then* $\overline{\gamma}(X_n) = 2$.

Proof : Let $n = 2^r, r \geq 2$. Then $U_n$consists all the odd vertices of $\mathbb{Z}_n$. Since $n$is even, then no two even labeled vertices are adjacent. This implies that even labeled vertices and odd labeled vertices form a bipartition of the vertex set. Now let D be a $\overline{\gamma}$-set of $X_n$and $i \in D$. Without loss of generality, let $i \in U_n$that is, i is an odd labeled vertex. Then i cannot be adjacent to any $k \in U_n \backslash \{i\}$ since $i - k \notin U_n$.Also,

$$\begin{aligned} deg(i) &= \phi(n) \\ &= 2^r - 2^{r-1} \\ &= 2^{r-1}(2-1) \\ &= 2^r \cdot 2^{-1} \\ &= \frac{2^r}{2} \\ &= \frac{n}{2}. \end{aligned}$$

Hence, i is adjacent to any $j \in \mathbb{Z}_n \backslash U_n$, that is, i dominates all the evenvertices of X_n. In a similar matter, j is adjacent to any$l \in U_n$, that is, $j$dominates all the odd vertices of X_n. Thus, $D = \{i, j\}$ and$D$ is a minimumclosed dominating set of X_n. Therefore, $\overline{\gamma}(X_n) = 2$. ∎

Theorem 3.4 *If* $n = 2p$*where p is prime, then* $\overline{\gamma}(X_n) = 2$.

Proof : Let $n = 2p$. Then $\phi(n) = p - 1$by Theorem 2.5. Since n is even, then due to Theorem 2.2, X_n, n≥ 2 is bipartite, i.e., even labeled vertices and the odd labeled vertices form a bipartition of the vertex set. Let D be a γ- set of X_n. Now, let i ∈D such that $i$is odd labeled vertex in X. Then $\deg(i) = p - 1$by Theorem 2.3. Since the number of even labeled vertices is p, then there exists an even labeled vertex say j, such that i and j are not adjacent. Thus i dominates all even labeled vertices except j. Similarly, we can show that j dominates all odd vertices except i. Thus, $D = \{i, j\}$ and D is a minimumdominating set of X_n. Therefore, $\overline{\gamma}(X_n) = |D| = 2$. ∎

Theorem 3.5 *If p and q are distinct odd primes and* $n = pq$, *then* $\overline{\gamma}(X_n) = p$,*where* $p < q$.

Proof : Let $n = pq$, where p and q are two distinct odd primes and $p < q$. Then

$$\begin{aligned} \phi(n) &= \phi(p)\phi(q) \\ &= (p-1)(q-1) \\ &= |U_n| \end{aligned}$$

|
by Theorem 2.6. Now, for $0 \leq i \leq p - 1$, we define each partite set as $A_i = \{x \in V(X_n): x \equiv i (mod\ p)\}$.Then $A_0 \cup \ldots \cup A_{p-1} = V(X_n)$ and $A_i \cap A_j = \emptyset, i \neq j, 1 \leq, j \leq i$, which implies that $A_i$forms an independent setbecause they are equivalence class modulo p. Thus, for any $v \in A_i$, $v$is not adjacent to $u \in A_i\{v\}$.Therefore, X_n forms a p- partition.

Let D be a γ-set of $X_n$and $v \in D$. Let $l_r = \{0, 1, \ldots, p - 1\}$. If $v \in A_i$ for any$0 \leq i \leq p - 1$,then there exists a $w_j \in A_j, j \neq i, 0 \leq j \leq p - 1$ such that $w_j = qr + v$ where $r \in l_r$, that is $\gcd(v - w, n) \neq 1$.Hence, there exists$w_j \in A_j$ such that$v$ and $w$are not adjacent. Thus, $w_j \in D$. Since, $j \neq i$and $0 \leq j \leq p - 1$, $|D| = p$and D is a minimum dominating set. Therefore,$\overline{\gamma}(X_n) = p$. ∎

Theorem 3.6 *If* $n = p^\alpha$*where p is prime,* $\alpha \geq 2$, *then* $\overline{\gamma}(X_n) = 2$.

Proof : Suppose $n = p^\alpha$where p is prime, $\alpha \geq 2$, and$v \in V(X_n)$. Then$\deg(v) = \phi(n) = p^\alpha - p^{\alpha-1}$.For $0 \leq i \leq p - 1$, we define each partite setas $A_i = \{x \in V(X_n): x \equiv i(mod\ p)\}$.Then $A_0 \cup \ldots \cup A_{p-1} = V(X_n)andA_i \cap A_j = \emptyset, i \neq j, 1 \leq, j \leq i$,which implies that $A_i$forms an independent setbecause they are equivalence class modulo p. Hence, for any $v \in A_i$, $v$is not adjacent to $u \in A_i\{v\}$.Therefore, $X_n$forms a partition.

Let D be a γ-set of $X_n$and $v \in D$. Without loss of generality, let $v \in A_i$for $0 \leq i \leq p - 1$. Since $\deg(v) = p^\alpha - p^{\alpha-1} = p^\alpha\left(1 - \frac{1}{p}\right)$,then v dominates$V(X_n) \backslash A_i$that is v is adjacent to any vertex $V(X)$ except for the elementsof $A_i$other than v itself. Take $u \in A_j$where $j \neq i\ and\ 0 \leq j \leq p - 1$. In thesame way, u dominates $V(X_n) \backslash A_j$and that includes A_i. Thus, $D = \{u, v\}$and$D$ is a minimum dominating set. Consequently, $\overline{\gamma}(X_n) = |D| = 2$. ∎

4. INVERSE CLOSED DOMINATION IN GRAPHS

Theorem 4.1 *If n is prime, then* $\bar{\gamma}^{-1}(X_n) = 1$.

Proof : Suppose n is prime. Then by Theorem 3.1 $\bar{\gamma}(X_n) = 1$, since the graph X_n is isomorphic to K_n with n vertices. Therefore by Theorem 2.7, $\bar{\gamma}^{-1}(X_n) = 1$. ■

Theorem 4.2 *Let* $n \geq 3$ *be an integer such that* $\phi(n) = 2$. *Then* $\bar{\gamma}^{-1}(X_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof : Suppose $\phi(n) = 2$. Then by the proof of Theorem 3.2, $X_n \cong C_n$, a cycle with n vertices. Since X_n is isomorphic to C_n, then by Theorem 2.8, we must have $\bar{\gamma}^{-1}(X_n) = \left\lceil \frac{n}{3} \right\rceil$. ■

Theorem 4.3 *If n is an integer such that* $n = 2^r, r \geq 2$, *then* $\bar{\gamma}^{-1}(X_n) = 2$.

Proof : Let $n = 2^r, r \geq 2$. Then U_n consists all the odd vertices of $\mathbb{Z}_n$. Since n is even, then no two even labeled vertices are adjacent. This implies that even labeled vertices and odd labeled vertices form a bipartition of the vertex set, i.e. X_n is isomorphic to $K_{\frac{n}{2},\frac{n}{2}}$. Hence, X_n is a complete bipartite graph. Therefore by Theorem 2.7, $\bar{\gamma}^{-1}(X_n) = 2$. ■

Remark 4.4 *Let* $n \geq 2$. *Then*

$$(i)\quad 1 \leq \bar{\gamma}^{-1}(X_n) < n;$$

$$(ii)\quad \gamma(X_n) \leq \bar{\gamma}^{-1}(X_n) \leq \gamma^{-1}(X_n).$$

Remark 4.5 *Let* $n \geq 2$. *Then* $\bar{\gamma}(X_n) \leq \bar{\gamma}^{-1}(X_n)$.
The *Join* of two graphs G and H is the graph $G + H$ with vertex-set $V(G + H) = V(G) \dot{\cup} V(H)$ and edge-set $E(G + H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$.

Clearly, $\bar{\gamma}^{-1}(X_n + K_1) = \bar{\gamma}(X_n)$. We consider $X_n + H$ with nontrivial graphs H. For any $u \in V(X_n)$ and $v \in V(H)$, the set $\{u, v\}$ is a closed dominating set in $X_n + H$. Thus, $\gamma(X_n + H) \leq 2$.

Lemma 4.6 *For nontrivial graph H and* $n \geq 2$, $\bar{\gamma}^{-1}(X_n + H) \leq 2$.

Proof : By the preceding remark, $\bar{\gamma}(X_n + H) \leq 2$. First, we consider the case where $\bar{\gamma}(X_n + H) = 1$, and suppose that $S = \{v\}$ is a closed dominating set in $X_n + H$. Assume $v \in V(X_n)$. Take $u \in V(X_n)\setminus\{v\}$ and $w \in V(H)$. Then $D = \{u, w\} \subseteq V(X_n + H)\setminus S$ and D is a closed dominating set in $X_n + H$. Thus $\bar{\gamma}^{-1}(X_n + H) \leq |D| = 2$. Next, we assume that $\bar{\gamma}(X_n + H) = 2$. Pick any $u \in V(X_n)$ and $v \in V(H)$. Then $S = \{u, v\}$ is a $\bar{\gamma}$*-set* in $X_n + H$. Thus, for any $x \in V(X)\setminus S$ and $y \in V(H)\setminus S$, the set $D = \{x, y\}$ is a $\bar{\gamma}^{-1}$*-set* in $X_n + H$. Since X_n and H are nontrivial graphs, such D exists. Thus $\bar{\gamma}^{-1}(X_n + H) = |D| = 2$. ■

Theorem 4.7 *Let H be a nontrivial graph and* $n \geq 2$. *If* $\bar{\gamma}^{-1}(X_n + H) = 1$, *then* $\bar{\gamma}(X_n) = 1$ *or* $\bar{\gamma}(H) = 1$. *The converse, however, is not necessarily true.*

Proof : The assumption implies that $\bar{\gamma}(X_n + H) = 1$. Therefore, $\bar{\gamma}(X_n) = 1$ *or* $\bar{\gamma}(H) = 1$.
To prove the second statement, consider the graph $K_{1,5} + P_7$. Note that $\bar{\gamma}(K_{1,5}) = 1$ but $\bar{\gamma}^{-1}(K_{1,5} + P_7) = 2$. ■

Theorem 4.8 *Let H be a nontrivial graph and* $n \geq 2$. *Then* $\bar{\gamma}^{-1}(X_n + H) = 1$ *if and only if one of the following is true:*

(i) $\bar{\gamma}(X_n) = 1$ *and* $\bar{\gamma}(H) = 1$;

(ii) $\bar{\gamma}(X_n) = 1$ and X_n *has at least two minimum* $\bar{\gamma}$*-sets;*

(iii) $\bar{\gamma}(H) = 1$ *and H has at least two minimum* $\bar{\gamma}$*-sets;*

Proof : Suppose that (i) holds and $\{v\} \subseteq V(X_n)$ and $\{w\} \subseteq V(H)$ are closed dominating sets in X_n and H, respectively. Then $\{v\}$ and $\{w\}$ are minimum closed dominating sets in $X_n + H$. The conclusion follows from the fact that since $\{v\} \subseteq V(X_n + H)\setminus\{w\}$, $\{v\}$ is a $\bar{\gamma}^{-1}$-set in X_n. Now, suppose that (ii) holds and let $\{u\}$ and $\{v\}$ be closed dominating sets in X_n. Then $\{u\}$ and $\{v\}$ are closed dominating sets in $X_n + H$. Since $\{u\} \subseteq V(X_n + H)\setminus\{v\}$, $\bar{\gamma}^{-1}(X_n + H) = 1$. Similarly, if (iii) holds, then $\bar{\gamma}^{-1}(X_n + H) = 1$. ■

Theorem 4.9 *Let $n \geq 2$. Then $\overline{\gamma}^{-1}(X_n + H) = 1$ if and only if $X_n = K_p,\ p \geq 2$, or $X_n = H + K$ for some nontrivial graphs H and K satisfying one ofthe following:*

$(i)\overline{\gamma}(H) = 1$ *and* $\overline{\gamma}(K) = 1$

$(ii)\overline{\gamma}(H) = 1$*and H has at least two minimum $\overline{\gamma}$-sets;*

$(iii)\overline{\gamma}(K) = 1$*and K has at least two minimum $\overline{\gamma}$-sets.*

Proof : First, note that $\overline{\gamma}^{-1}(K_p) = 1$ for all $p \geq 2$. Suppose that $X_n$is anoncomplete graph. Suppose, further, that$\overline{\gamma}^{-1}(K_p) = 1$. Then there exist two distinct vertices u and v of $X_n$such that $\{u\}$ and $\{v\}$ are $\overline{\gamma}$-*sets* in X_n. Moreover,$uv \in E(G)$.Put $H = \langle\{u, v\}\rangle$and $K = \langle X_n - \{u, v\}\rangle$. Then $X_n = H + K$.Furthermore, $\{u\}$ and$\{v\}$are two distinct $\overline{\gamma}$-*sets* in H. Consequently, (ii) holds. The converse follows immediately from Theorem 4.8. ■

Theorem 4.10 *Let H be a nontrivial graph and $n \geq 2$. Then $\overline{\gamma}^{-1}(X_n + H) = 2$if and only if any of the following is true:*

$$(i)\,\overline{\gamma}(X_n) \geq 2 \text{ and } \overline{\gamma}(H) \geq 2$$

$(ii)\overline{\gamma}(X_n) = 1$ *and* $\overline{\gamma}(H) \geq 2$*but*$X_n \neq K_1 + (K_1 + \cup_j X_{n_j})$ *for any graphs* X_{n_j}.

Proof : Suppose that $\overline{\gamma}^{-1}(X_n + H) = 2$. Then either $\overline{\gamma}(X_n + H) = 1$ *or* $\overline{\gamma}(X_n + H) = 2$.
It is clear that if $\overline{\gamma}(X_n + H) = 2$then $\overline{\gamma}(X_n) \geq 2$and $\overline{\gamma}(H) \geq 2$. Suppose that$\overline{\gamma}^{-1}(X_n + H) = 1$. Then $\overline{\gamma}(X_n) = 1$ or $\overline{\gamma}(H) = 1$.Assume that $\overline{\gamma}(X_n) = 1$.Then $X_n = \{v\} + \cup_j X_{n_j})$for some components $X_{n_j}$of X_n. Thus,

$$\overline{\gamma}^{-1}(X_n + H) = \overline{\gamma}\left(H + \cup_j X_{n_j}\right) = 2.$$

Necessarily, $\overline{\gamma}(H) \geq 2$ and $\overline{\gamma}\left(\cup_j X_{n_j}\right) \geq 2$.This means that, in particular,
$X_n \neq K_1 + (K_1 + \cup_j X_{n_j})$.

To prove the converse, we first consider the case where $\overline{\gamma}(X_n) \geq 2$ and $\overline{\gamma}(H) \geq 2$.Then
$\overline{\gamma}(X_n + H) = 2$. Since $\overline{\gamma}(X_n + H) \leq \overline{\gamma}^{-1}(X_n + H)$, then $\overline{\gamma}^{-1}(X_n + H) \geq 2$. Now pick $u \in V(X_n)$ and $v \in V(H)$, and let $x \in V(X)\backslash\{u\}$ and $y \in V(H)\{v\}$. Then $S = \{u, v\}$is a minimum dominating set in $X_n + H$so that$D = \{x, y\}$is a $\overline{\gamma}^{-1}$-set in $X_n + H$..Thus $\overline{\gamma}^{-1}(X_n + H) \leq 2$. Accordingly, $\overline{\gamma}^{-1} = 2$.
Next, we proceed with the case where $\overline{\gamma}(X_n) = 1$ and $\overline{\gamma}(H) \geq 2$but $X_n \neq K_1 + \left(K_1 + \cup_j X_{n_j}\right)$.Let$S = \{u\} \subseteq V(X_n)$be a closed dominating setin X_n. Then S is a closed dominating set in $X_n + H$. We consider

$$(X_n + H) - u = (X_n - u) + H$$

The condition for $X_n$implies that $X_n - u \neq K_1 + U_j X_n$for any componentsof X_{n_j} of X_n.Thus, $\overline{\gamma}(X_n - u) \geq 2$. If $\overline{\gamma}(X_n - u) \geq 2$ and $\overline{\gamma}(H) \geq 2$, then $\overline{\gamma}^{-1}(X_n + H) = \overline{\gamma}\big((X_n - u)\big) = 2$. ■

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