Volume 5, No.5, May 2018
Journal of Global Research in Mathematical Archives
UGC Approved Journal
RESEARCH PAPER
Available online at http://www.jgrma.info

ISSN 2320 – 5822

Domination in Cube of Graphs

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Abstract: Let $G = (V, E)$ be a graph. The cube of a graph G is denoted by G^3 has the same vertex set as in G and every two vertices $u, v \in V(G^3)$ are adjacent in G^3 if and only if they are joined by a path of length ≤ 3. In this paper, we establish the bounds for the domination number of cube of G in terms of G. Also we investigate their relationship with other domination parameters.

Keywords: domination; square of a graph; cube of a graph.
AMS Subject Classification: 05C69.

1 Introduction

All graphs considered throughout this paper are simple and nontrivial. For undefined terms or notations may be found in [4]. Let $G = (V, E)$ be a graph. The number of vertices of G we denote by n and the number of edges we denote by m, thus $|V(G)| = n$ and $|E(G)| = m$. By the *open neighborhood* of a vertex v of G we mean the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. By the *closed neighborhood* of a vertex v of G we mean the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree of a vertex* v, denoted by $deg(v)$, is the cardinality of its open neighborhood. A vertex is called *isolated* if it has no neighbors, while it is called *universal* if it is adjacent to all other vertices. Let $\Delta(G)(\Delta')$ mean the maximum degree(edge) among all vertices(edges) of G. The path (cycle, respectively) on n vertices we denote by P_n (C_n, respectively). A wheel W_n, where $n \geq 4$, is a graph with n vertices, formed by connecting a vertex to all vertices of a cycle C_{n-1}. By $K_{p,q}$ we denote a complete bipartite graph with partite sets of cardinalities p and q.

2 Motivation

M. A. Henning et al.[7] gave an interpretation for the k-distance domination and domination of the power of a graph G, such as $\gamma_k(G) = \gamma(G^k)$, for any connected G. F. Harary et al. [3] defined squares of graph and using this M. H. Muddebihal et al.[8] used the notation for 2-distance domination $\gamma_2(G)$ as $\gamma(G^2)$ and called it as domination in G^2 of a graph G. Motivated by this we defined cubes of graphs and used the notation for 3-distance domination $\gamma_3(G)$ *as* $\gamma(G^3)$ and called it as domination in G^3 of a graph G.

Since the distance version of domination have a strong background of applications in interconnection networks, many efforts have been made by several authors to consider the distance parameters. The diameter of the graph defined as the maximum distance between any two vertices, represents the maximum communication link between any two vertices of the network. As a consequence, when finding interconnection network models it is important for the diameter to be small as possible.

In this paper we establish bounds for the domination number of cubes of graphs.

3 Preliminaries

We now define domination and its related parameters which have been studied by different mathematicians. These parameters are of great practical interest because of the applications of domination theory in different fields.

A set D of vertices in a graph G is a *dominating set* if every vertex in $V-D$ is adjacent to some vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A minimum dominating set of a graph G is called a $\gamma-$ *set* of G.

- A *connected dominating set* is a dominating set in which $\langle D\rangle$ is connected.
- A dominating set D is *independent dominating set* if $\langle D\rangle$ is independent.
- A dominating set D is called a *total dominating set* if there are no isolates in $\langle D\rangle$.
- A dominating set D is called a *split dominating set* if $\langle V-D\rangle$ is disconnected.

The minimum cardinality taken over all connected \ independent \ total \ split \ nonsplit \ cototal \ maximal gives the respective domination numbers $\gamma_c(G)$ \ $\gamma_i(G)$ \ $\gamma_t(G)$ \ $\gamma_s(G)$ respectively.

The *distance* $d(u,v)$ between two vertices u and v in G is the length of a shortest path joining them if any; otherwise $d(u,v)=\infty$. A shortest $u-v$ path is called a *geodesic*. The *diameter* of a connected graph G is the length of any longest geodesic and is denoted as $diam(G)$. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is $max\ d(u,v)$ for all u in G. The *radius* $r(G)$ is the minimum eccentricity of the points. The maximum eccentricity is the *diameter*. A point v is a *central point* if $e(v)=r(G)$ and the *center* of G is the set of all central vertices. The *girth* of a graph G denoted by $g(G)$, is the length of a shortest cycle in G. For more details about domination and distance concepts interested readers can refer [1, 2, 5, 6].

Definition 1. 1 *Let $G=(V,E)$ be a graph. The cube of a graph G is denoted by G^3 has the same vertex set as in G and every two vertices $u,v\in V(G^3)$ are adjacent in G^3 if and only if they are joined by a path of length one or two or three.*

4 Results

Since we are considering all connected graphs with diameter at least three. Therefore the domination number of cube of complete graph, wheel graph, star graph, complete bipartite graph does not make any sense. Therefore from the list of standard class of graphs we are considering only path graph ($P_n; n\geq 4$) and cycle graph ($C_n; n\geq 6$).

Observation 1. 2 *For any graph G, $\gamma(G^2)\leq\gamma(G)$ and $\gamma(G^3)\leq\gamma(G^2)$.*

The following result gives the exact values of domination number of cube of path and cycle graph respectively.

Proposition 2. 3 *If $G = P_n$ or C_n, then $\gamma(P_n^3) = \left\lceil \frac{n}{7} \right\rceil = \gamma(C_n^3)$.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices of a path P_n. By the definition of cubes of a graph v_1 is adjacent to v_2, v_3 and v_4 respectively, and v_2 is adjacent to v_1, v_3, v_4 and v_5 and so on. In general every v_i is adjacent to $v_{i-1}, v_{i-2}, v_{i-3}, v_{i+1}, v_{i+2}$ and v_{i+3} only when $i \geq 2$. Now we construct a vertex for G^3 as follows: $X = \{v_4, v_8, v_{12}, \cdots v_{4i}\}$ where $1 \leq i \leq \left\lceil \frac{n}{7} \right\rceil$. The above set X is a dominating set for G^3 because every vertex not in X is adjacent to a vertex in X. Moreover for each vertex $v_i \in X$ there exist a vertex $u_i \in V - X$ such that $N(u_i) \cap X = \{v\}$. Now each vertex in the set X is of maximum degree in G^3 and the vertices 4_i for $1 \leq i \leq \left\lceil \frac{n}{7} \right\rceil$. being nonadjacent to each other will dominate maximum distinct vertices of G^3. Therefore the set X is of minimum cardinality. Hence $\gamma(G^3) = |X| = \left\lceil \frac{n}{7} \right\rceil$. . Hence, the domination number of cubes of path graph P_n is $\left\lceil \frac{n}{7} \right\rceil$. i.e $\gamma(P_n)^3 = \left\lceil \frac{n}{7} \right\rceil$. .

The proof for cubes of cycle graph C_n follows the same lines of the proof of cubes graph of path graph P_n.

Theorem 3. 4 *For any graph G, $\gamma(G^3) = 1$ if and only if there exist a central vertex $v \in V(G)$ such that $e(v) \leq 3$.*

Proof. Let G be any graph with $diam(G) \geq 3$. Let D be a minimal dominating set of G^3. If $\gamma(G^3) = 1$ and G contains a central vertex $v \in G$ such that $e(v) \geq 4$. Then there exist a vertex $u \in V \backslash D$ such that u is not dominated by any vertex in D. Hence $D' = D \cup \{u\}$ will form a minimal dominating set for G^3, which is a contradiction to the minimality of D. Hence G must contain a vertex v such that $e(v) \leq 3$.

Theorem 4. 5 *If H is a spanning subgraph of G then $\gamma(G) \leq \gamma(H)$ and $\gamma(G^3) \leq \gamma(H^3)$.*

Proof. It is well known fact that if H is a spanning subgraph of G then $\gamma(G) \leq \gamma(H)$. Now the only part we have to prove is H^3 is spanning subgraph of G^3. Since H is spanning subgraph of G therefore by the definition of G^3, H^3 must be a spanning subgraph of G^3. Hence, $\gamma(G^3) \leq \gamma(H^3)$.

5 Bound for $\gamma(G^3)$

First, we obtain an upper bound for $\gamma(G^3)$ in terms of order and maximum degree $\Delta(G^3)$.

Theorem 5. 6 *For any connected graph G, $\gamma(G^3) \leq n - \Delta(G^3)$.*

Proof. Let G be any connected graph with $diam(G) = k$ where $k \geq 3$. Since $\gamma(G^3) \leq \gamma(G)$. Therefore, the result follows from Observation 1.

Next, we obtain an upper bound for $\gamma(G^3)$ in terms of vertex connectivity $\kappa(G^3)$.

Theorem 6. 7 *For any graph G, with $n \geq 4$,*

$$\gamma(G^3) \leq n - \kappa(G^3).$$

Proof. Since for disconnected graph G, we cannot define G^3. So we shall assume that G is connected graph of order at least four with $diam(G) \geq 3$. Which implies that G is non-complete graph. Let $\{v_1, v_2, \cdots, v_k\} = F$ be the set of vertices whose removal results in a disconnected graph of G^3. That is F is a minimum vertex cut of G^3 (i.e., $\kappa(G^3) = |F|$). Next we have to show that $V(G^3) - F$ is a dominating set of G. Suppose this is not true then there exists a vertex $v_i \in F; 1 \leq i \leq k$ such that $N(v_i)$ and $V - F$ are disjoint. Also $v_i \in F - N(v_i)$ which implies that $N(v_i)$ is a proper subset of F. So $|F| \leq n - 2$ and $V(G^3) - F$ has at least one element. Therefore the subgraph induced by the set $V(G^3) - N(v_i)$ is disconnected. Therefore the removal of neighborhood vertices of v_i results in a disconnected graph of G^3, that is $N(v_i)$ is the minimal vertex cut of G^3. But F is a minimal vertex cut of G^3 which is a contradiction to the minimality F. Therefore, $V(G^3) - F$ is a dominating set G^3. Thus,

$$\begin{aligned} \gamma(G^3) &\leq |V(G^3) - F| \\ &\leq n - \kappa(G^3) \end{aligned}$$

.

Theorem 7. 8 *Let G^3 be a cube graph of a graph H of order n such that $\gamma(G^3) = n - \kappa(G^3)$. Then G^3 is regular graph.*

Proof. By Theorem 5, we have $\gamma(G^3) \leq n - \Delta(G^3)$ and since $\gamma(G^3) = n - \kappa(G^3)$. We have $n\Delta(G^3) \geq n - \kappa(G^3)$. Which gives, $\Delta(G^3) \geq \kappa(G^3)$. Since, $\kappa(G^3) \leq \delta(G^3)$ therefore $\Delta(G^3) \leq \delta(G^3)$ and $\delta(G^3) \leq \Delta(G^3)$ is always true. Hence G^3 is a regular graph.

Theorem 8. *If G^3 is hamiltonian, then $\gamma(G^3) \leq \left\lceil \frac{n}{7} \right\rceil$.*

Proof. Let G be any connected graph of order at least four with $diam(G) \geq 3$. Since G^3 is hamiltonian therefore G^3 contains a spanning cycle H such that $\gamma(H) = \left\lceil \frac{n}{3} \right\rceil$. By Observation 1, we have

$$\begin{aligned} \gamma(G^3) &\leq \gamma(G) \\ &\leq \gamma(H) \\ &\leq \left\lceil \frac{n}{3} \right\rceil. \end{aligned}$$

Next, we find the lower bound for cube of a graph G, in terms of diameter of a graph G.

Theorem 9. 9 *For any connected graph G, $\left\lceil \frac{diam(G)+1}{7} \right\rceil \leq \gamma(G^3)$. Further equality holds for $G = P_n$.*

Proof. Let G be any connected graph with $diam(G) \geq 3$ and let $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$ and $E(G) = \{e_1, e_2, e_3, \cdots, e_m\}$. By the definition of cubes of graphs every vertex $v_i \in V(G)$ is adjacent to its neighbors and all the vertices at a distance 2 and 3 respectively. Clearly the degree of each pendant vertex in G^3 is 3. i.e, $deg_{G^3}(v_i) = 3$ and degrees of all support vertices will be 4. Finally the degree of remaining vertices will be at least 5. Let $D = \{v_1, v_2, \cdots, v_k\}$ for $1 \leq k \leq \left\lceil \frac{n}{7} \right\rceil$ be a dominating set of G^3. Consider an arbitrary path of length $diam(G)$ and this diametral path induces at most six edges from the induced subgraph $\langle N[v] \rangle$ for each $v \in D$ because $deg_D(v_i) \geq 6$. Furthermore, the diametral path induces at most $\gamma(G^3) - 1$ edges joining the neighborhoods of the vertices of D. Hence

$$diam(G) \leq 6\gamma(G^3)+\gamma(G^3)-1$$
$$\left\lceil\frac{diam\,(G)+1}{7}\right\rceil \leq \gamma(G^3).$$

If $G = P_n$, then we know that $diam(P_n) = n-1$ and by Proposition 2, result follows.

In the following Theorems 5-11, we establish different upper bounds for $\gamma(G^3)$.

Theorem 10. 10 *For any connected graph G, $\gamma(G^3) \leq \left\lceil\frac{n}{7}\right\rceil$. Further, equality holds if $G = P_n$ or C_n.*

Proof. Let G be any graph of order n and size m and let cubes of a graph G be G^3. Let D be a dominating set of G^3. Then we have to show that the cardinality of D must not exceed $\leq \left\lceil\frac{n}{7}\right\rceil$. To show this, we have to choose the vertices in G which are adjacent to at least six vertices in G^3. consider an induced path P_7 say X_1 in which the vertices are labeled as $v_1v_2v_3v_4v_5v_6v_7$ in G. Here we have to select v_4 which is adjacent at least six vertices in G^3. Therefore $v_4 \in D$. Similarly select another edge disjoint induced path P_7 say X_2 in G with labeled as $v_1'v_2'v_3'v_4'v_5'v_6'v_7'$ here we have to choose v_7' to the set D. Continuing this process for all induced paths $X_i's$ in G and choose a vertex v_{4i} to the set D.

Hence $D = \{v_4, v_{4'}, \cdots, v_{4i'}\}$ for $1 \leq i \leq \left\lceil\frac{n}{7}\right\rceil$ will form a dominating set for G^3. Further, note that set X is of maximum degree and G^3 and the vertices $4i$ for $1 \leq i \leq \left\lceil\frac{n}{7}\right\rceil$ being nonadjacent to each other will dominate maximum distinct vertices of G^3. Therefore the set X is of minimum cardinality.

Thus

$$\gamma(G^3) = |D|$$
$$= |\{v_{4i}\}|, \; 1 \leq i \leq \left\lceil\frac{n}{7}\right\rceil$$
$$\leq \left\lceil\frac{n}{7}\right\rceil.$$

Suppose $G = P_n$ or C_n, then the result follows from Proposition 2.

Corollary 11. 11 *Let G be any connected graph. Suppose $diam(G) = k$ for some positive integer k, then $\gamma(G^3) \leq \left\lceil\frac{k}{7}\right\rceil$.*

Theorem 12. 12 *For any connected graph G with $\delta(G) \geq 2$, $\gamma(G^3) \leq \left\lceil\frac{n-\Delta(G)+2}{7}\right\rceil$. Further equality holds for $G = C_n$.*

Proof. Let G be any (n,m) −graph with diameter is at least 3. Then $\Delta(G) \leq n-2$. Let v be a vertex of maximum degree in G, then v must be a maximum degree vertex in G^3. Then v is adjacent to $N(v), N(N(v))$ and $N(N(N(v)))$ vertices. Such that $\Delta(G^3) = |N(v)| \cup |N(N(v))| \cup |N(N(N(v)))|$. Hence $V(G^3) - \Delta(G^3)$ will form a dominating set for G^3. Further, note that the $\delta(G) \geq 2$, so every vertex in G must be in some edge disjoint induced path P_7. Also, we know that $\delta(G) \leq \Delta(G)$. Therefore, combining all these properties of G and by Theorem 7, we get $\gamma(G^3) \leq \left\lceil\frac{n-\Delta(G)+2}{7}\right\rceil$.

For equality, suppose G is a cycle C_n then the $\delta(G) = \Delta(G) = 2$ by using Proposition 2, we get the required result.

Theorem 13. 13 *If every support vertex of a tree* T *is adjacent to at least one pendant vertex, then* $\gamma(T^3) \leq \left\lceil \frac{n-r}{4} \right\rceil$, *where r is the number of pendant vertices in* T.

Proof. Let $X = \{v_1, v_2, \ldots, v_k\}$ be the set of all support vertices and $Y = \{u_1, u_2, \ldots, u_r\}$ be the set of pendant vertices such that $X \cup Y = V$. Since each vertex of Y is adjacent with at least one vertex of X in T, we have X itself a dominating set of T. Now, in graph $T^3, B = \{v_1, v_2, \ldots, v_i\}$, $i \leq k$, denotes the dominating set of T^3 such that $B \subseteq X$ and every pendant vertex $u_i, i = 1,2, \ldots, r$ in G forms C_3 *and* C_4 with at least two vertices in X i.e.,

$$\begin{aligned} B &\subseteq \frac{X}{4} \\ |B| &\leq \frac{|V|-|Y|}{4} \\ &\leq \frac{n-r}{4} \\ \gamma(T^3) &\leq \left\lceil \frac{n-r}{4} \right\rceil. \end{aligned}$$

Hence the proof.

Theorem 14. 14 *For any connected graph* G , $\gamma(G^3) \leq \left\lceil \frac{n-\alpha_0(G)}{2} \right\rceil$.

Proof. Let $X = \{v_1, v_2, \ldots, v_m\}$ where $deg(v_i) \geq 2,\ 1 \leq i \leq m$ be the set of vertices which covers all the edges of G, such that $|X| = \alpha_0(G)$. Now in G^3, let $D = \{v_1, v_2, \ldots, v_k\}$ be the minimal dominating set. Since $V(G^3) = V(G)$ it follows that $D \subseteq \frac{X}{2}$. Clearly,

$$|D| \leq \frac{n-|X|}{2}$$

and hence

$$\gamma(G^3) \leq \left\lceil \frac{n-\alpha_0(G)}{2} \right\rceil.$$

We need the following theorem for our next result.

Theorem 15. 15 *[4] For any graph* G, $\chi(G) \leq \Delta(G) + 1$. *Where* $\chi(G)$ *is the chromatic number of a graph* G.

Theorem 16. 16 *For any connected graph* G, $\gamma(G^3) + \chi(G \left\lceil \frac{n+7}{7} \right\rceil + \Delta(G)$. *Where* $\chi(G)$ *is the chromatic number of a graph* G. *Further equality holds for* $G = C_{2k+1}$, *for any positive integer* k.

Proof. Let G be any connected graph and let $X = \{x_1, x_2, \cdots, x_i\}$ for some $1 \leq i \leq n$ be any proper coloring of G. By combining the results of Theorem 7 we have $\gamma(G^3) \leq \left\lceil \frac{n+7}{7} \right\rceil$ and Brook's Theorem, $\chi(G) \leq \Delta(G) + 1$. Hence

$$\begin{aligned} \gamma(G^3) + \chi(G) &\leq \left\lceil \frac{n+7}{7} \right\rceil + \Delta(G) + 1 \\ &\leq \left\lceil \frac{n+7}{7} \right\rceil + \Delta(G). \end{aligned}$$

For equality, suppose $G = C_{2k+1}$, then by Proposition 4, $\gamma(C^3) = \left\lceil\frac{n+7}{7}\right\rceil$ and we know that $\chi(C_{2k+1}) = \Delta(C_{2k+1}) + 1$. Hence by combining these two facts, we get the equality for the above bound.

6 Relation with other domination parameters

Theorem 17. **17***For any graph* G*,*

$$\left\lceil\frac{\gamma(G^3)+\gamma(G)}{2}\right\rceil \le \gamma_t(G).$$

Proof. To prove this result we consider the following two cases.

case 1: Suppose G is a tree, $X = \{v_1, v_2, \ldots, v_k\}$ be the set of all pendant vertices of G and $V' = V - X$, then $D' \subseteq V'$ is a minimal dominating set of G. Further, $D' \cup H$, where $H \in N(D')$ and $H \subseteq V(G) - D'$ forms minimal total dominating set of G. If $v_j = \{\emptyset\}$, then there exist at least one vertex $v \in X$ such that $D' \cup \{v\}$ forms a total dominating set of G^3. Let $D = \{u_1, u_2, \ldots, u_m\}$ be the dominating set of G^3. If the neighbors of each u_i, $1 \le i \le m$ are at a distance of at most three which generates D to be a minimal dominating set of G^3. Then we have,

$$\left\lceil\frac{|D|+|D'|}{2}\right\rceil \le |D' \cup H|.$$

Gives the required result.

case 2: Suppose G is not a tree and let D be a minimal dominating set of G and $V' = V - D$. Further, $H \in N(D')$, such that $H \subseteq V'$. Now $D' \cup H$ forms minimal total dominating set of G. In G^3 let $D = \{u_1, u_2, \ldots, u_m\}$ be minimal dominating set which is also subset of D'. Now

$$\left\lceil\frac{|D|+|D'|}{2}\right\rceil \le |D' \cup H|$$

gives the required result.

Theorem 18. 18 *For any graph* G, $\gamma(G^3) < \gamma_c(G)$.

Proof. Let P be a diametral path such that $P\colon v_1, v_2, \ldots, v_k$. Suppose $I = \{v_1, v_2, \ldots, v_i\}, i \le k$ connected dominating set of G such that $I \subseteq V(P)$ so that $|I| = \gamma_c(G)$. Now we consider $D = \{v_{i-1}, v_{i-2}, \ldots, v_{i-m}\}$, $m \le i$ forms dominating set of G^3 and if each neighbor of $v_{i-m}, m \le i$ is at a distance of at most three forms D to be a minimal dominating set. Hence $|D| = \gamma(G^3)$. Clearly, $\gamma(G^3) < \gamma_c(G)$.

Finally we establish Nordhaus-Gaddum type result.

Theorem 19. 19 *For any connected graph* G*,*

(i). $\gamma(G^3) + \gamma(\overline{G}^3) \le \left\lceil\frac{2n}{7}\right\rceil$ and

(ii). $\gamma(G^3) \cdot \gamma(\overline{G}^3) \le \left\lceil\frac{n}{7}\right\rceil$

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