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SOME PARAMETERS ON EQUITABLE COLORING OF PRISM AND CIRCULANT GRAPH.

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Abstract

Coloring the vertices of a graph G according to certain condition is a random experiment and a discrete random variable X is defined as the number of vertices having a particular color in the given type of coloring of G and a probability mass function for this random variable can be defined accordingly. In this paper we extend the concepts of arithmetic mean and variance to the theory of equitable graph coloring and determine the values of these parameters.

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1 Introduction

All graphs considered in this paper are finite, loopless and without multiple edges. A graph $G = (V(G), E(G))$ is said to be equitably k-colorable if the vertex set $V(G)$ can be partitioned into k independent subsets $V_1, V_2, ..., V_K$ such that $||V_i| - |V_j|| \leq 1$ for all i and j. Each V_i is said to form a color class. The smallest integer k for which G is equitably k-colorable is called the equitable chromatic number of G. W. Meyer [9] introduced the notion of equitable colorability.

Equitable Coloring Conjecture (ECC) [9] For any connected graph G, which is neither a complete graph nor an odd cycle,$\chi_e(G) \leq \Delta(G)$, where $\Delta(G)$ is the maximum vertex degree in G.

Sudev et.al [10] have found the mean and variance of equitable coloring of certain graphs. K.P.Chithra et.al [5] found the equitable coloring parameters of certain wheel related graphs. Sudha et.al [11, 12] have discussed the total coloring of Prism and $X(Z_n, C)$ graph. For a proper k-coloring $\{c_1, c_2, ..., c_k\}$ of G, we can define a random variable $(r.v)X$ which denotes the color of any arbitrary vertex in G. As the sum of all weights of colors of G is the order of G, the real valued function $f(i)$ defined by

$$f(i) = \begin{cases} \frac{\theta(c_i)}{|V(G)|}, & \text{if } i = 1, 2, 3, ..., k \\ 0, & \text{otherwise} \end{cases}$$

is the probability mass function (p.m.f) of the random variable $(r.v)X$

Definition 1.1. *Let $\{c_1, c_2, ..., c_k\}$ be a certain type of proper k-coloring of a given graph G and $f(i)$ denotes the p.m.f of a particular color c_i assigned to the vertices of G. Then,*
1. *The coloring mean of a coloring of a given graph G, denoted by $\mu_c(G)$ or simply $\mu(G)$, is defined to be $\mu_c(G) = E(i) = \sum\limits_{i=1}^{k} i.f(i)$ and*
2. *The coloring variance of is defined as $\sigma_c^2(G) = V(i) = \sum\limits_{i=1}^{k} (i - \mu)^2 f(i)$.*

1.1 Equitable Chromatic Parameters of Graphs

An equitable coloring of a graph G is a proper coloring of G with an assignment of colors to the vertices of G such that the number of vertices in any two color classes differ by at most one. The equitable chromatic number of a graph G is the smallest number k such that G has an equitable coloring with k colors.

Definition 1.2. *The coloring mean of a graph G with respect to a proper coloring is said to be an equitable chromatic mean or χ_e-chromatic mean of G if it is the minimum equitable coloring of G and the coloring mean is also minimum. The χ_e-chromatic mean of a graph G is denoted by $\mu_{\chi_e}(G)$*

Definition 1.3. *The χ_e-chromatic variance of G denoted by $\sigma^2_{\chi_e}$ is a coloring variance of G with respect to a minimal equitable coloring of G which yields the minimum coloring sum.*

Definition 1.4. *A coloring mean of a graph G, with respect to a proper coloring is said to be a χ_e^+-chromatic mean of G if it is the minimum equitable colouring of G, but the colouring mean is maximum. The χ_e^+-chromatic mean of a graph G is denoted by $\mu_{\chi_e^+}(G)$.*

Definition 1.5. *The χ_e^+-chromatic variance of G denoted by $\sigma^2_{\chi_e^+}(G)$, is a coloring variance of G with respect to a minimal equitable coloring of G which yields the maximum coloring sum.*

Definition 1.6. *A Prism Y_n is the Cartesian product of the cycle C_n and the path P_n.*

Definition 1.7. [6] *Let Z_n denote the additive group of integers modulo n. If C is a subset of $Z_n \setminus 0$, then construct a graph $X = X(Z_n, C)$ as follows. The vertices of X are the elements of Z_n and (i, j) is an arc of X if and only if $j - i \in C$. The graph $X(Z_n, C)$ is called a circulant of order n, and C is called its connection set.*

Since $X(Z_n, C)$-graph satisfies the conditions required for an Hamiltonian and an Eulerian, it is both Hamiltonian and Eulerian. Here the connection set C is $\{\pm 1, \pm 2\}$

Equitable coloring is being used in scheduling problems in which jobs have to be allocated to workers such that the maximum difference between any two workers in the allocation is one.In this paper we find the χ_e-chromatic mean, χ_e-chromatic variance, χ_e^+-chromatic mean, χ_e^+-chromatic variance of Prism Y_n and $S(n, 2)$-graph.

2 The χ_e and χ_e^+ -chromatic parameters of Prism

Theorem 2.1. *The χ_e-chromatic mean of the prism Y_n is*

$$\mu_{\chi_e}(Y_n) = \begin{cases} \frac{3}{2}, & if\ n\ is\ even \\ 2, & if\ n\ is\ odd\ and\ n \equiv 0 (mod\ 3) \\ \frac{4n-1}{2n}, & if\ n\ is\ odd\ and\ n \equiv 1, 2 (mod\ 3) \end{cases}$$

and the χ_e-chromatic variance of the prism Y_n is

$$\sigma^2_{\chi_e}(Y_n) = \begin{cases} \frac{1}{4}, & if\ n\ is\ even \\ \frac{2}{3}, & if\ n\ is\ odd\ and\ n \equiv 0 (mod\ 3) \\ \frac{8n^2-2n-3}{12n^2}, & if\ n\ is\ odd\ and\ n \equiv 1 (mod\ 3) \\ \frac{8n^2+2n-3}{12n^2}, & if\ n\ is\ odd\ and\ n \equiv 2 (mod\ 3) \end{cases}$$

Proof. Let $\{v_i,\ 1 \le i \le n\}$ be the vertex set of Y_n. Suppose n is even. Then Y_n is bipartite and can be colored with 2-colors c_1 and c_2 which will be a minimal equitable coloring of Y_n. Then,$\mu_{\chi_e}(Y_n) = \frac{3}{2}$ and $\sigma^2_{\chi_e}(Y_n) = \frac{1}{4}$.

Let n be an odd integer. Then any proper coloring of Y_n must contain atleast three colors say c_1, c_2, and c_3. Let C_1, C_2 and C_3 be the respective color classes. Then we have the following three cases.

Case 1: $n \equiv 0(mod\ 3)$

Color the vertices of Y_n in such a way that $C_1 = \{v_i : i \equiv 0(mod\ 3)\}$, $C_2 = \{v_i : i \equiv 1(mod\ 3)\}$ and $C_3 = \{v_i : i \equiv 2(mod\ 3)\}$. Then each color class will have exactly $\frac{n}{3}$ vertices and hence is a minimal equitable coloring of Y_n with its p.m.f defined as

$$f(i) = \begin{cases} \frac{1}{3}, & i = 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

Hence the parametrs are $\mu_{\chi_e}(Y_n) = 2$ and $\sigma^2_{\chi_e}(Y_n) = \frac{2}{3}$.

Case 2: $n \equiv 1(mod\ 3)$

In this case C_1 contains $\frac{2n-1}{3}$ vertices and the remaining two color classes contain $\frac{2n+1}{3}$ vertices each. Hence the corresponding p.m.f is defined as,

$$f(i) = \begin{cases} \frac{2(n-1)}{3n}, & i = 3 \\ \frac{2n+1}{6n}, & i = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e}(Y_n) = \frac{4n-1}{2n}$ and $\sigma^2_{\chi_e}(Y_n) = \frac{8n^2-2n-3}{12n^2}$.

Case 3: $n \equiv 2(mod\ 3)$

In this case C_1 contains $\frac{2n-1}{3}$ vertices and the remaining two color classes contain $\frac{2n+2}{3}$ vertices each. Hence the corresponding p.m.f is defined as,

$$f(i) = \begin{cases} \frac{2n-1}{6n}, & i = 2,3 \\ \frac{2n+1}{6n}, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e}(Y_n) = \frac{4n-1}{2n}$ and $\sigma^2_{\chi_e}(Y_n) = \frac{8n^2+2n-3}{12n^2}$. ■

Theorem 2.2. *The χ_e^+-chromatic mean of the prism Y_n is*

$$\mu_{\chi_e^+}(Y_n) = \begin{cases} \dfrac{3}{2}, & if\ n\ is\ even \\ 2, & if\ n\ is\ odd\ and\ n \equiv 0(mod\ 3) \\ \dfrac{4n+1}{2n}, & if\ n\ is\ odd\ and\ n \equiv 1,2(mod\ 3) \end{cases}$$

and the χ_e^+*-chromatic variance of the prism* Y_n *is*

$$\sigma^2_{\chi_e^+}(Y_n) = \begin{cases} \frac{1}{4}, & if\ n\ is\ even \\ \frac{2}{3}, & if\ n\ is\ odd\ and\ n \equiv 0(mod\ 3) \\ \frac{8n^2-2n-3}{12n^2}, & if\ n\ is\ odd\ and\ n \equiv 1(mod\ 3) \\ \frac{8n^2+2n-3}{12n^2}, & if\ n\ is\ odd\ and\ n \equiv 2(mod\ 3) \end{cases}$$

Proof. For the cases when n is even and n is odd with $n \equiv 0(mod\ 3)$ all the coloring classes have the same number of vertices. Hence, reversing the coloring pattern will also give the same mean and variance.

That is if n is even then $\mu_{\chi_e^+}(Y_n) = \frac{3}{2}$ and $\sigma^2_{\chi_e^+}(Y_n) = \frac{1}{4}$.

Also when n is odd and $n \equiv 0(mod\ 3)$, $\mu_{\chi_e^+}(Y_n) = 2$ and $\sigma^2_{\chi_e^+}(Y_n) = \frac{2}{3}$.

When n is odd and $n \equiv 1(mod\ 3)$ reversing the coloring pattern we have

$$f(i) = \begin{cases} \frac{n-1}{3n}, & i = 1 \\ \frac{2n+1}{6n}, & i = 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e^+}(Y_n) = \frac{4n+1}{2n}$ and $\sigma^2_{\chi_e^+}(Y_n) = \frac{8n^2+2n-3}{12n^2}$

Similarly by reversing the cloring pattern for the case n is odd and $n \equiv 2(mod\ 3)$

$$f(i) = \begin{cases} \frac{2n-1}{6n}, & i = 1, 2 \\ \frac{2n+2}{6n}, & i = 3 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e^+}(Y_n) = \frac{4n+1}{2n}$ and $\sigma^2_{\chi_e^+}(Y_n) = \frac{8n^2+2n-3}{12n^2}$.

3 The χ_e and χ_e^+ chromatic parameters of the $X(Z_n, C)$ Graphs

Theorem 3.1. *The* χ_e*-chromatic mean of* $X(Z_n, C)$ *is*

$$\mu_{\chi_e X(Z_n,C)} = \begin{cases} 2, & if\ n \equiv 0(mod\ 3) \\ \frac{5}{2}, & if\ n \equiv 0(mod\ 4) \\ \frac{5n-3}{2n}, & if\ n \equiv 1, 3(mod\ 4) \\ \frac{5n-4}{2n}, & if\ n \equiv 2(mod\ 4) \end{cases}$$

and the χ_e-chromatic variance of $X(Z_n, C)$ is

$$\sigma^2_{\chi_e} X(Z_n, C) = \begin{cases} \frac{2}{3}, & if\ n \equiv 0(mod\ 3) \\ \frac{5}{4}, & if\ n \equiv 0(mod\ 4) \\ \frac{5n^2+4n-9}{4n^2}, & if\ n \equiv 1(mod\ 4) \\ \frac{5n^2-16}{12n^2}, & if\ n \equiv 2(mod\ 4) \\ \frac{5n^2-4n-9}{4n^2}, & if\ n \equiv 3(mod\ 4) \end{cases}$$

Proof. In $X(Z_n, C)$ has the maximum degree of a vertex is 4. Hence any proper equitable coloring may have at most 4 colors.

Case 1: $n \equiv 0(mod\ 3)$

In this case the graph has an equitable coloring with three colors and hence each color class will have$\frac{n}{3}$ vertices. By defining the p.m.f

$$f(i) = \begin{cases} \frac{1}{3}, & i = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

we get $\mu_{\chi_e^+}(X(Z_n, C)) = 2$ and $\sigma^2_{\chi_e^+}(X(Z_n, C)) = \frac{2}{3}$

Case 2: $n \equiv 0(mod\ 4)$

In this case the vertex set can be split up into four color classes each having $\frac{n}{4}$ vertices. By defining the p.m.f

$$f(i) = \begin{cases} \frac{1}{4}, & i = 1, 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

we get $\mu_{\chi_e}(X(Z_n, C)) = \frac{5}{2}$ and $\sigma^2_{\chi_e}(X(Z_n, C)) = \frac{5}{4}$

Case 3: $n \equiv 1(mod\ 4)$

The first color class will have $\frac{n-1}{4}$ vertices and the remaining three classes will have $\frac{n+3}{4}$ each .Therefore the p.m.f is given by

$$f(i) = \begin{cases} \frac{n-1}{4n}, & i = 2, 3, 4 \\ \frac{n+3}{4n}, & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

which yields $\mu_{\chi_e}(X(Z_n, C)) = \frac{5n-3}{2n}$ and $\sigma^2_{\chi_e}(X(Z_n, C)) = \frac{5n^2+4n-9}{4n^2}$

Case 4: $n \equiv 2(mod\ 4)$

In this case the two color classes will have $\frac{n-2}{4}$ vertices and the remaining three classes will have $\frac{n+2}{4}$ each. Define the p.m.f as

$$f(i) = \begin{cases} \frac{n-2}{4n}, & i = 3,4 \\ \frac{n+2}{4n}, & i = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

which yields $\mu_{\chi_e}(X(Z_n, C)) = \frac{5n-4}{2n}$ and $\sigma^2_{\chi_e}(X(Z_n, C)) = \frac{5n^2-16}{4n^2}$

Case 5: $n \equiv 3(mod\ 4)$

In this case the first color classes will have $\frac{n-3}{4}$ vertices and the remaining three classes will have $\frac{n+1}{4}$ each .The p.m.f is given by

$$f(i) = \begin{cases} \frac{n-3}{4n}, & i = 3,4 \\ \frac{n+1}{4n}, & i = 1,2 \\ 0, & \text{otherwise} \end{cases}$$

which yields $\mu_{\chi_e}(X(Z_n, C)) = \frac{5n-3}{2n}$ and $\sigma^2_{\chi_e}(X(Z_n, C)) = \frac{5n^2-4n-9}{4n^2}$ ■

Theorem 3.2. *The χ_e^+-chromatic mean of $X(Z_n, C)$ is*

$$\mu_{\chi_e^+} X(Z_n, C) = \begin{cases} 2, & if\ n \equiv 0(mod\ 3) \\ \frac{5}{2}, & if\ n \equiv 0(mod\ 4) \\ \frac{5n+3}{2n}, & if\ n \equiv 1,3(mod\ 4) \\ \frac{5n+4}{2n}, & if\ n \equiv 2(mod\ 4) \end{cases}$$

and the χ_e^+-chromatic variance of $X(Z_n, C)$ is

$$\sigma^2_{\chi_e^+}(X(Z_n, C)) = \begin{cases} \frac{2}{3}, & if\ n \equiv 0(mod\ 3) \\ \frac{5}{4}, & if\ n \equiv 0(mod\ 4) \\ \frac{5n^2+4n-9}{4n^2}, & if\ n \equiv 1(mod\ 4) \\ \frac{5n^2-16}{12n^2}, & if\ n \equiv 2(mod\ 4) \\ \frac{5n^2-4n-9}{4n^2}, & if\ n \equiv 3(mod\ 4) \end{cases}$$

Proof. For the cases with $n \equiv 0(mod\ 3)$ and $n \equiv 0(mod\ 4)$ all the coloring classes have the same number of vertices. Hence reversing the coloring pattern will also give the same mean and variance. Hence $\mu_{\chi_e^+}(X(Z_n, C)) = 2$ and $\sigma^2_{\chi_e^+}(X(Z_n, C)) = \frac{2}{3}$ for $n \equiv 0(mod\ 3)$
$\mu_{\chi_e^+}(X(Z_n, C)) = \frac{5}{2}$ and $\sigma^2_{\chi_e^+}(X(Z_n, C)) = \frac{5}{4}$ for $n \equiv 0(mod\ 4)$
By reversing the coloring pattern for the case $n \equiv 1(mod\ 4)$ the pmf is

$$f(i) = \begin{cases} \frac{n-1}{4n}, & i = 1, 2, 3 \\ \frac{n+3}{4n}, & i = 4 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e^+}(X(Z_n, C)) = \frac{5n+3}{2n}$ and $\sigma^2_{\chi_e^+}(X(Z_n, C)) = \frac{5n^2+4n-9}{4n^2}$
For the case $n \equiv 2(mod\ 4)$ the p.m.f is

$$f(i) = \begin{cases} \frac{n-2}{4n}, & i = 1, 2 \\ \frac{n+2}{4n}, & i = 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e^+}(X(Z_n, C)) = \frac{5n+4}{2n}$ and $\sigma^2_{\chi_e^+}(X(Z_n, C)) = \frac{5n^2-16}{4n^2}$
For the case $n \equiv 3(mod\ 4)$ the p.m.f is

$$f(i) = \begin{cases} \frac{n-3}{4n}, & i = 1 \\ \frac{n+1}{4n}, & i = 2, 3, 4 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\mu_{\chi_e^+}(X(Z_n, C)) = \frac{5n+3}{2n}$ and $\sigma^2_{\chi_e^+}(X(Z_n, C)) = \frac{5n^2-4n-9}{4n^2}$ ■

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