RANDOM SOLUTION OF SECOND ORDER RANDOM PERIODIC BOUNDARY VALUE PROBLEM

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Abstract: We discuss the existence and multiplicity of random positive solutions for the second order random periodic boundary value problem using Guo-Krasnosel’skii fixed point theorem.

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1. INTRODUCTION

Consider the second order random periodic boundary value problem

\[ u''(t,\omega) = f(t,u(t,\omega),\omega), \quad \text{a.e. } t \in [0,2\pi], \]
\[ u(0,\omega) = u(2\pi,\omega), \]
\[ u'(0,\omega) = u'(2\pi,\omega). \]

Where \( f \) is random caratheodory function. By introducing two height functions concerned with \( f \) and considering integrals of height functions on some bounded sets.

We discuss the existence and multiplicity of positive random solutions for the second order random periodic boundary value problem (1.1) using Guo-Krasnosel’skii fixed point theorem. The random positive solution of problem (1.1) means a solution \( u \) of (1.1) satisfying \( u(t,\omega) > 0, \text{ a.e. } t \in [0,2\pi], \omega \in \Omega \), and

\[ f : J \times R \times \Omega \rightarrow R. \text{ Assume that } k \text{ is a constant satisfying } 0 < k < \frac{1}{4} \text{ and } \]

\[ f : [0,2\pi] \times [-\infty,\infty] \times \Omega \rightarrow [-\infty,\infty] \text{ is a random caratheodory function.} \]

Definition 1.1. Random caratheodory: The function \( f \) is said to be random caratheodory if satisfies the following conditions:

1. The function \( f(t,\omega,\cdot) : (0,\infty) \times \Omega \rightarrow (-\infty,\infty) \) is continuous, for a.e. \( t \in [0,2\pi] \),
2. The function \( f(\cdot,u,\omega) : [0,2\pi] \times \Omega \rightarrow (-\infty,\infty) \) is measurable, for all \( u \in (0,\infty) \),
3. For any \( 0 < c < d \), there exists a non-negative function \( h(c,d) \in L[0,2\pi] \) such that

\[ |f(t,u,\omega)| \leq h(c,d)(t,\omega), \text{ a.e. } t \in [0,2\pi] \text{ and } \forall u \in [c,d], \omega \in \Omega. \]

To the best of our knowledge, the second order random periodic boundary value problem (1.1) has not been discussed earlier in the literature. But the special case, when the random parameter is absent from the
problem \((1.1)\), we obtain a classical second order periodic boundary value problem \((1.1)\) and have been investigated by many authors [1–6]. In most real problems, only the positive solution is significant. Recently, Torres [5] proved the following existence theorem.

**Theorem 1.1:** Let there exist two positive numbers \(a, b\) such that

\[
(1) . \ f(t,u) + ku \geq 0, \ a.e. \ t \in [0,2\pi], \ \omega \in \Omega, \ \forall \ u \in [\sigma \min \{a,b\}, \max \{a,b\}].
\]

\[
(2) . \ f(t,u) + ku \leq \frac{1}{2\pi M} u, \ a.e. \ t \in [0,2\pi], \ \omega \in \Omega, \ \forall \ u \in [\sigma a,a].
\]

\[
(3) . \ f(t,u) + ku \geq \frac{1}{2\pi m} u, \ a.e. \ t \in [0,2\pi], \ \omega \in \Omega, \ \forall \ u \in [\sigma b,b].
\]

Then problem \((1.1)\) has one random positive solution.

Here, \(m = \frac{\cos \sqrt{k} \pi}{2\sqrt{k} \sin \sqrt{k} \pi}, \ M = \frac{1}{2\sqrt{k} \sin \sqrt{k} \pi}, \ \sigma = \cos \sqrt{k} \pi < 1.\)

Remark: Theorem 1.1 is an effective tool for the positive solution of problem \((1.1)\) if \(f(t,u)/u\) are essential bounded in \(t\) on bounded closed sets \([0,2\pi] \times [\sigma a,a]\) and \([0,2\pi] \times [\sigma b,b]\). But Theorem 1.1 is powerless if \(f(t,u)/u\) is not essential bounded in \(t\) on one of these sets.

The aim of this paper is improve Theorem 1.1. The idea of this work comes from Torres [5], Jiang [6], Qingliu Yao [7–12]. We will draw into two height functions to describe the growth feature of term \(f\). And then, we will apply uo-Krasnosel’skii fixed point theorem of cone expansion–compression type to establish a basic existence theorem. The basic existence theorem shows that the problem \((1.1)\) has at least one random positive solution provided that the integrals of height functions are appropriate on some bounded sets. In Section 3, we will also consider the existence of multiple random positive solutions.

### 2. AUXILIARY RESULTS

Consider the Banach space \(C[0,2\pi]\) with norm \(\|u\| = \max_{0 \leq t \leq 2\pi} |u(t,\omega)|\) and let

\[C^+[0,2\pi] = \{u \in C[0,2\pi] : u(t,\omega) \geq 0, 0 \leq t \leq 2\pi\},\]

\[K = \{u \in C^+[0,2\pi] : u(t,\omega) \geq \sigma u, 0 \leq t \leq 2\pi\}.\]

Let \(\Omega(c) = \{u \in K : u < c\}, \ \partial \Omega(c) = \{u \in K : u = c\}.\)

Denote

\[
G(t,s,\omega) = \frac{\cos \sqrt{k} (\pi - t + s + \omega)}{2\sqrt{k} \sin \sqrt{k} \pi}, \quad 0 \leq s \leq t \leq \omega \leq 2\pi
\]

\[
G(t,s,\omega) = \frac{\cos \sqrt{k} (\pi + t - s + \omega)}{2\sqrt{k} \sin \sqrt{k} \pi}, \quad 0 \leq t \leq s \leq \omega \leq 2\pi.
\]

Since \(G(t,s,\omega) > 0, \ 0 \leq t, s, \omega \leq 2\pi, \ \omega \in \Omega\). We obtain

\[m = \min_{0 \leq t, s \leq 2\pi} G(t,s,\omega), \ M = \max_{0 \leq t, s \leq 2\pi} G(t,s,\omega), \ \sigma = mM^{-1}.\]

Define the random operator \(T\) as follows
\((Tu)(t,\omega) = \int_{0}^{2\pi} G(t,s,\omega)[f(s,u(s,\omega),\omega)+ku(s,\omega)] \, ds, 0 \leq t \leq 2\pi \text{ and } \omega \in \Omega.\)

**Theorem 2.1:** (1) For any \(0 < r_{1} < r_{2}\), \(T: \overline{\Omega(r_{2})} \setminus \Omega(r_{1}) \rightarrow C[0,2\pi]\) is a random compact operator.
(2) If \(u \in K\) such that \(f(t,u(t,\omega),\omega)+ku(t,\omega) \geq 0\), a.e. \(t \in [0,2\pi]\), then \(Tu \in K\).
(3) If \(u^{*} \in K\) is a fixed point of \(T\) and \(u^{*} \neq 0\), then \(u^{*}\) is a random positive solution of (1.1).

Proof: (1) The compactness of the random operator \(T\) on \(\overline{\Omega(r_{2})} \setminus \Omega(r_{1})\) can be proved by a standard argument.

(2) If \(f(t,u(t,\omega),\omega)+ku(t,\omega) \geq 0\), a.e. \(t \in [0,2\pi]\), then,

\[ (Tu)(t,\omega) = \int_{0}^{2\pi} G(t,s,\omega)[f(s,u(s,\omega),\omega)+ku(s,\omega)] \, ds \]

\[ \geq \int_{0}^{2\pi} m[f(s,u(s,\omega),\omega)+ku(s,\omega)] \, ds \]

\[ = mM^{-\pi} \int_{0}^{2\pi} M[f(s,u(s,\omega),\omega)+ku(s,\omega)] \, ds \]

\[ \geq \sigma \max_{0 \leq s \leq 2\pi} \int_{0}^{2\pi} G(t,s,\omega)[f(s,u(s,\omega),\omega)+ku(s,\omega)] \, ds \]

\[ = \sigma\|Tu\|. \]

Thus, \(Tu \in K\).

(3) If \(u^{*} \neq 0\) is a fixed point of \(T\), then \(u^{*} > 0\) and \(u^{*}(t,\omega) \geq u^{*} > 0, 0 \leq t \leq 2\pi\).

Clearly, the problem (1.1) is equivalent to the problem (1.2), here

\[ u^{*}(t,\omega)+ku^{*}(t,\omega) = f(t,u^{*}(t,\omega),\omega)+ku^{*}(t,\omega) \text{ a.e. } t \in [0,2\pi], \]

\[ u(0,\omega) = u(2\pi,\omega), \quad u'(0,\omega) = u'(2\pi,\omega). \]

(2.1)

Using this, the problem (2.1) is equivalent to the integral equation

\[ u(t,\omega) = \int_{0}^{2\pi} G(t,s,\omega)[f(s,u(s,\omega),\omega)+ku(s,\omega)] \, ds, 0 \leq t \leq 2\pi. \]

In other words, the random fixed point \(u^{*}\) of \(T\) is a random positive solution of (2.1).

We will apply the following Guo-Krasnosel’skii random fixed point theorem to find nonzero random fixed point of \(T\) in \(K\).

**Theorem 2.2:** Let \(X\) be a Banach space, \(K\) be a cone in \(X\), \(\Omega_{1}, \Omega_{2}\) be two bounded open sets in \(K\) such that \(0 \in \Omega_{1}, \overline{\Omega_{1}} \subset \Omega_{2}\). If \(T: \overline{\Omega_{2}} \setminus \Omega_{1} \rightarrow K\) is a random compact operator and one of the following conditions is satisfied:

(1) \(\|Tu(t,\omega)\| \leq \|u(t,\omega)\|\), \(u \in \partial \Omega_{1}\) and \(\|Tu(t,\omega)\| \geq \|u(t,\omega)\|\), \(u \in \partial \Omega_{2}\).

(2) \(\|Tu(t,\omega)\| \geq \|u(t,\omega)\|\), \(u \in \partial \Omega_{1}\) and \(\|Tu(t,\omega)\| \leq \|u(t,\omega)\|\), \(u \in \partial \Omega_{2}\).

Then \(T\) has a random fixed point in \(\overline{\Omega_{2}} \setminus \Omega_{1}\).
We draw into the following height functions:

\[ \varphi(t, r, \omega) = \sup \{ f(t, u, \omega) + k u : u \in [\sigma r, r] \}, \]
\[ \psi(t, r, \omega) = \inf \{ f(t, u, \omega) + k u : u \in [\sigma r, r] \}. \]

In geometry, \( \phi(t, r, \omega) \) and \( \psi(t, r, \omega) \) express maximal height and minimal height of function \( f(t, u, \omega) + k u \) on the segment \( \{t, \omega\} \times [\sigma r, r] \), respectively.

According to (3) of caratheodory function, for any \( r > 0 \), there exists a non-negative function \( h_{(\sigma r, r)} \in L[0, 2\pi] \) such that

\[ 0 \leq \psi(t, r, \omega) \leq \varphi(t, r, \omega) \leq h_{(\sigma r, r)}(t, \omega) + k r, \text{ a.e. } t \in [0, 2\pi]. \]

Thus, \( \psi(\cdot, r, \omega), \varphi(\cdot, r, \omega) \in L[0, 2\pi] \).

3. Existence of single and multiplicity of positive solution

We quote the following basic existence theorem.

**Theorem 3.1**: Assume that there exist two positive numbers \( a, b \) such that

(1). \( f(t, u, \omega) + k u \geq 0, \text{ a.e. } t \in [0, 2\pi], \forall u \in [\sigma \min \{a, b\}, \max \{a, b\}] \).

(2). \( \int_0^{2\pi} \varphi(t, a, \omega) dt \leq a M^{-1} \) and \( \int_0^{2\pi} \psi(t, b, \omega) dt \geq b M^{-1} \).

Then problem (1.1) has at least one random positive solution \( u^* \in K \) satisfying \( \min \{a, b\} \leq u^* \leq \max \{a, b\} \).

**Proof**: It is easy to see \( a \neq b \). Without loss of generality, let \( a < b \).

By the condition (1) and Theorem 2.1, (1)–(2), we see that the random operator \( T : \overline{\Omega}(b) \setminus \Omega(a) \to K \) is random compact.

If \( u \in \partial \Omega(a) \), then \( \sigma a \leq u(t, \omega) \leq a, \ 0 \leq t \leq 2\pi \).

By (1) and (2),

\[ 0 \leq f(t, u(t, \omega), \omega) + k u(t, \omega) \leq \varphi(t, a, \omega), \ 0 \leq t \leq 2\pi. \]

It follows,

\[ \|Tu\| = \max_{0 \leq s \leq 2\pi} \int_0^{2\pi} G(t, s, \omega) \left[ f(s, u(s, \omega), \omega) + k u(s, \omega) \right] ds \]
\[ \leq \max_{0 \leq s \leq 2\pi} \int_0^{2\pi} G(t, s, \omega) \varphi(s, a, \omega) ds \]
\[ \leq M \int_0^{2\pi} \varphi(s, a, \omega) ds \]
\[ = M a M^{-1} = a = \|u\|. \]

If \( u \in \partial \Omega(b) \), then \( \sigma b \leq u(t, \omega) \leq b, \ 0 \leq t \leq 2\pi. \)

By By (1) and (2),

\[ f(t, u(t, \omega), \omega) + k u(t, \omega) \geq \psi(t, b, \omega) \geq 0, \ 0 \leq t \leq 2\pi. \]

Thus,

\[ \|Tu\| = \max_{0 \leq s \leq 2\pi} \int_0^{2\pi} G(t, s, \omega) \left[ f(s, u(s, \omega), \omega) + k u(s, \omega) \right] ds \]
By Theorem 2.2, we assert that \( T \) has at least one random fixed point \( u^* \in \Omega(b) \setminus \Omega(a) \).

Thus \( u^* \in K \) and \( a \leq \|u^*\| \leq b \). By Theorem 2.1(3), \( u^* \) is a random positive solution of (1.1).

Let \([c]\) be the integer part of \( c \). The following result concerns the existence of \( n \) positive random solutions.

**Theorem 3.2:** Assume that there exist \( n+1 \) positive numbers \( a_i < a_2 < \cdots < a_{n+1} \) such that

1. \( f(t, u, \omega) + ku \geq 0, \ a.e. \ t \in [0, 2\pi], \ \forall u \in [\sigma a_i, a_{n+1}] \).

2. One of following conditions is satisfied:

   - (2-A). \( \int_0^{2\pi} \varphi(t, a_{2i-1}, \omega) dt < a_{2i-1} M^{-1}, \ i = 1, 2, \ldots, \left[ \frac{n+2}{2} \right] \),
   
   and \( \int_0^{2\pi} \psi(t, a_i, \omega) dt > a_i m^{-1}, \ i = 1, 2, \ldots, \left[ \frac{n+1}{2} \right] \).

   - (2-B). \( \int_0^{2\pi} \psi(t, a_{2i-1}, \omega) dt > a_{2i-1} m^{-1}, \ i = 1, 2, \ldots, \left[ \frac{n+2}{2} \right] \),
   
   and \( \int_0^{2\pi} \varphi(t, a_i, \omega) dt > a_i M^{-1}, \ i = 1, 2, \ldots, \left[ \frac{n+1}{2} \right] \).

Then problem (1.1) has at least \( n \) random positive solutions \( u_i^* \in K, \ i = 1, 2, \ldots, n \) satisfying \( a_i < \|u_i^*\| < a_{i+1} \).

**Proof:** From Theorem 3.1, we can prove that if there exist two positive numbers \( a, b \) such that

\[
\int_0^{2\pi} \varphi(t, a, \omega) dt < a M^{-1} \quad \text{and} \quad \int_0^{2\pi} \psi(t, b, \omega) dt > b m^{-1},
\]

then problem (1.1) has at least one random positive solution \( u^* \in K \) satisfying \( \min \{a, b\} < \|u^*\| < \max \{a, b\} \).

By the claim, for every pair of positive numbers \( \{a_i, a_{i+1}\}, \ i = 1, 2, \ldots, n \), (1.1) has one random positive solution \( u_i^* \in K \) satisfying \( a_i < \|u_i^*\| < a_{i+1} \).

**4. Existence Results**

We assume that there exist the following limit functions

\[
F_\infty(t, \omega) = \lim_{u \to +\infty} \frac{f(t, u, \omega)}{u}, \quad F_0(t, \omega) = \lim_{u \to 0^+} \frac{f(t, u, \omega)}{u},
\]

\[
\phi_\infty(t, \omega) = \lim_{r \to +\infty} \frac{\varphi(t, r, \omega)}{r}, \quad \phi_0(t, \omega) = \lim_{r \to 0^+} \frac{\varphi(t, r, \omega)}{r},
\]

\[
\Psi_\infty(t, \omega) = \lim_{r \to +\infty} \frac{\psi(t, r, \omega)}{r}, \quad \Psi_0(t, \omega) = \lim_{r \to 0^+} \frac{\psi(t, r, \omega)}{r},
\]

Since the limit functions are measurable in \([0, 2\pi]\).

The following conditions are applied:

(i). \( f(t, u, \omega) + ku \geq 0, \ a.e. \ t \in [0, 2\pi], \ \forall u \in (0, +\infty) \).
(ii). There exist a non-negative function $q \in L[0, 2\pi]$ and constants $0 < c_1 < c_2 < +\infty$ such that
$$f(t,u,\omega)/u \leq q(t,\omega), \quad \text{a.e. } t \in [0, 2\pi], \quad \forall u \in (0,c_1] \cup [c_2, +\infty).$$

**Theorem 4.1:** If the condition (ii) holds, then

$$\lim_{u \to 0^+} \int_0^{2\pi} \left[ f(t,u,\omega)/u \right] dt = \int_0^{2\pi} F_0(t,\omega) dt, \quad \lim_{u \to +\infty} \int_0^{2\pi} \left[ f(t,u,\omega)/u \right] dt = \int_0^{2\pi} F_\infty(t,\omega) dt,$$

$$\lim_{r \to +\infty} \int_0^{2\pi} \left[ \varphi(t,u,\omega)/u \right] dt = \int_0^{2\pi} \phi_0(t,\omega) dt, \quad \lim_{r \to +\infty} \int_0^{2\pi} \left[ \varphi(t,u,\omega)/u \right] dt = \int_0^{2\pi} \phi_\infty(t,\omega) dt,$$

$$\lim_{r \to +\infty} \int_0^{2\pi} \left[ \psi(t,u,\omega)/u \right] dt = \int_0^{2\pi} \Psi_0(t,\omega) dt, \quad \lim_{r \to +\infty} \int_0^{2\pi} \left[ \psi(t,u,\omega)/u \right] dt = \int_0^{2\pi} \Psi_\infty(t,\omega) dt,$$

Proof: By the condition (ii), there exists a non-negative function $q \in L[0, 2\pi]$ and a constant $c_2 > 0$ such that
$$f(t,u,\omega)/u \leq q(t,\omega), \quad \text{a.e. } t \in [0, 2\pi], \quad \forall u \in [c_2, +\infty).$$

If $\sigma r \geq c_2$, then for a.e. $t \in [0, 2\pi]$, $\forall u \in [c_2, r],
$$f(t,u,\omega) + ku/r \leq f(t,u,\omega)/u + k \leq q(t,\omega) + k.$$ It follows $\psi(t,r,\omega)/r \leq \varphi(t,r)/r \leq q(t)/r$, a.e. $t \in [0, 2\pi]$. By Lebesgue dominated convergence theorem, we get

$$\lim_{u \to 0^+} \int_0^{2\pi} \left[ f(t,u,\omega)/u \right] dt = \int_0^{2\pi} F_0(t,\omega) dt,$$

$$\lim_{r \to +\infty} \int_0^{2\pi} \left[ \varphi(t,r,\omega)/r \right] dt = \int_0^{2\pi} \phi_0(t,\omega) dt,$$

$$\lim_{r \to +\infty} \int_0^{2\pi} \left[ \psi(t,r,\omega)/r \right] dt = \int_0^{2\pi} \Psi_0(t,\omega) dt,$$

**Theorem 4.2:** Assume that the condition (i) holds.

(1) $\int_0^{2\pi} F_0(t,\omega) dt < M^{-1} - 2\pi k$, then $\int_0^{2\pi} \phi_0(t,\omega) dt < M^{-1}$.

2) $\int_0^{2\pi} F_\infty(t,\omega) dt < M^{-1} - 2\pi k$, then $\int_0^{2\pi} \phi_\infty(t,\omega) dt < M^{-1}$.

3) $\int_0^{2\pi} F_0(t,\omega) dt > \sigma^{-1}m^{-1} - 2\pi k$, then $\int_0^{2\pi} \Psi_0(t,\omega) dt > m^{-1}$.

4) $\int_0^{2\pi} F_\infty(t,\omega) dt > \sigma^{-1}m^{-1} - 2\pi k$, then $\int_0^{2\pi} \Psi_\infty(t,\omega) dt > m^{-1}$.

Proof: For (1). We have
$$F_0(t,\omega) + k = \lim_{u \to 0^+} \sup_{0 < u \leq \omega} f(t,u,\omega)/u + k = \lim_{r \to +\infty} \sup_{0 < u \leq \omega} f(t,u,\omega)/u + k$$
$$= \lim_{r \to +\infty} \sup_{0 < u \leq \omega} [f(t,u,\omega)/u + ku/r] \geq \lim_{r \to +\infty} \sup_{0 < u \leq \omega} f(t,u,\omega)/u + k$$
\[
\geq \lim_{r \to 0+} \sup \left\{ f(t,u,\omega) + ku \right\} / u = \lim_{r \to 0} \phi(t,r,\omega) / r = \phi_0(t).
\]

Integrating this inequality from 0 to 2\(\pi\), we obtain
\[
\int_0^{2\pi} \phi_0(t,\omega) dt \leq \int_0^{2\pi} F_0(t,\omega) dt + 2\pi k < M^{-1} - 2\pi k + 2\pi k = M^{-1}.
\]

For (4), we have
\[
F_\infty(t,\omega) + k = \liminf_{u \to 0+} f(t,u,\omega) / u = \liminf_{r \to 0+} f(t,u,\omega) / u + k
\leq \liminf_{r \to 0+} f(t,u,\omega) / u \leq \liminf_{r \to 0+} f(t,u,\omega) / u + ku / r
\]
\[
= \liminf_{r \to 0+} f(t,u,\omega) / (\sigma r) = \lim_{r \to 0+} \psi(t,r,\omega) / r = \sigma^{-1}\Psi_\infty(t,\omega).
\]

Thus, \(\Psi_\infty(t,\omega) \geq \sigma \left( F_\infty(t,\omega) + k \right)\) and
\[
\int_0^{2\pi} \Psi_\infty(t,\omega) dt \geq \sigma \left( \int_0^{2\pi} F_\infty(t,\omega) dt + 2\pi k \right) > \sigma \sigma^{-1} m^{-1} = m^{-1}.
\]

Similarly proved (2) and (3).

5. On the limit cases

Using Theorems 4.1 and 4.2, we can prove the following existence and multiplicity results concerning limit cases.

Theorem 5.1: Assume that the conditions (i) and (ii) of above holds and one of the following conditions is satisfied:

1. \(\int_0^{2\pi} \phi_0(t,\omega) dt < M^{-1}\) and \(\int_0^{2\pi} \Psi_\infty(t,\omega) dt > m^{-1}\).

2. \(\int_0^{2\pi} \phi_\infty(t,\omega) dt < M^{-1}\) and \(\int_0^{2\pi} \Psi_0(t,\omega) dt > m^{-1}\).

3. \(\int_0^{2\pi} F_0(t,\omega) dt < M^{-1} - 2\pi k\), and \(\int_0^{2\pi} F_\infty(t,\omega) dt > \sigma^{-1} m^{-1} - 2\pi k\).

4. \(\int_0^{2\pi} F_\infty(t,\omega) dt < M^{-1} - 2\pi k\) and \(\int_0^{2\pi} F_0(t,\omega) dt > \sigma^{-1} m^{-1} - 2\pi k\).

Then problem (1.1) has at least one random positive solution \(u^* \in K\).

Proof: First of all, by the condition (ii) and Theorem 4.1 we have
\[
\lim_{r \to 0+} \int_0^{2\pi} \left[ \phi(t,r,\omega) / r \right] dt = \int_0^{2\pi} \phi_0(t,\omega) dt < M^{-1},
\]
\[
\lim_{r \to 0+} \int_0^{2\pi} \left[ \psi(t,r,\omega) / r \right] dt = \int_0^{2\pi} \Psi_\infty(t,\omega) dt < m^{-1}.
\]
It follows that there exist positive numbers \( 0 < a < b < +\infty \) such that 
\[
\int_0^{2\pi} \phi(t, a, \omega) \, dt < aM^{-1}
\]
and 
\[
\int_0^{2\pi} \psi(t, b, \omega) \, dt < bM^{-1}
\]  By (1) and Theorem 3.1, we assert that the problem (1.1) has at least one random positive solution \( u^* \in K \).

Similarly proved, (2) is similar to (1). and (3) and (4) are from (1) and (2) by applying Theorem 4.2, respectively.

**Example**: Consider the random periodic boundary value problem
\[
\begin{align*}
\begin{split}
\phi''(t, \omega) &= q(t, \omega) \max \left\{ \frac{2u(t, \omega)}{3\sqrt{3}}, 4u(t, \omega) - 10 \right\} - \frac{1}{9}u(t, \omega), \ \ a.e. \quad t \in [0, 2\pi] \\
\phi(0, \omega) &= \phi(2\pi, \omega), \quad \phi'(0, \omega) = \phi'(2\pi, \omega)
\end{split}
\end{align*}
\]
(A)

where \( q(t, \omega) = \frac{1}{\pi - 2} \left[ \frac{1}{\sqrt{t(2\pi - t)}} - \frac{1}{\pi} \right] \).

Thus, 
\[
\begin{align*}
\phi''(t, \omega) + ku &= q(t, \omega) \max \left\{ \frac{2u}{3\sqrt{3}}, 4u - 10 \right\},
\end{align*}
\]

Which gives 
\[
\int_0^{2\pi} q(t, \omega) \, dt = \frac{1}{\pi - 2} \left[ \int_0^{2\pi} \frac{1}{\sqrt{t(2\pi - t)}} \, dt - \int_0^{2\pi} \frac{1}{\pi} \, dt \right] = 1.
\]

It is easy to see that 
\[
\begin{align*}
\phi_0(t, \omega) &= \frac{2}{3\sqrt{3}} q(t, \omega), \quad \Psi_\infty(t, \omega) = 2q(t, \omega).
\end{align*}
\]

Thus, 
\[
\begin{align*}
\int_0^{2\pi} \phi_0(t, \omega) \, dt &= \frac{2}{3\sqrt{3}} < \frac{1}{\sqrt{3}} = M^{-1}. \\
\int_0^{2\pi} \Psi_\infty(t, \omega) \, dt &= 2 > \frac{2}{\sqrt{3}} = m^{-1}.
\end{align*}
\]

By Theorem 5.1 (1), the problem (A) has one random positive solution.

**REFERENCES**


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