DYNAMICAL SYSTEMS INDUCED BY COMPOSITION OPERATORS IN SPACES OF ANALYTIC FUNCTIONS

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Abstract. In this article, we study a composition operator $C_t$ which induce dynamical systems on various spaces like weighted Bergman space $\mathcal{A}_p^\alpha$, weighted Lebesgue space $L_p^w$, weighted Hardy space $H_p^w$, classical Dirichlet space $H(D)$, VMOA space and little Bloch space. Some examples of dynamical systems on different spaces are also studied.

1. Introduction

Let $D$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $X$ be a Banach space of analytic functions, analytic on $D$. A one parameter semigroup of analytic functions is any continuous homomorphism $\Phi : t \to \phi(t) = \phi_t$ from the additive semigroup of non negative real numbers into the composition semigroup of all analytic functions which maps $D$ into $D$ and for which the following three conditions hold.

(i) $\phi_0$ is the identity in $D$
(ii) $\phi_{t+s} = \phi_t \circ \phi_s$, for all $t, s \geq 0$
(iii) $\phi_t \to \phi_0$, as $t \to 0$, uniformly on compact subsets of $D$

The condition (iii) can be replaced by
Each such semigroup gives rise to a semigroup $C_t$ consisting of composition operators such that $C_t : X \to X$ defined by $C_t(f) = f \circ \phi_t$ for $t \geq 0$.

2. Preliminaries

Definition 2.1. Dynamical system [5]:
Let $G$ be a topological group with $e$ as the identity. Let $X$ be a topological space and $\pi : G \times X \to X$ be a continuous map such that
(i) $\pi(e, x) = x$ for every $x \in X$
(ii) $\pi(st, x) = \pi(s, \pi(t, x))$ for every $s, t \in G$ and $x \in X$
The transformation group $(Z, X, \pi)$ is called a discrete dynamical system.

Definition 2.2. Weighted Bergman space $A^p_\alpha$ [1]:
For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha$ consists of the functions $f$ analytic on the unit disk $D$ such that
$$\|f\|_{p, \alpha}^p = \frac{1}{\pi} \int_0^1 M_p(r, f)(1 - r^2)^\alpha r dr < \infty$$
where $M_p(r, f) = \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$

Definition 2.3. Lebesgue dominated convergence theorem [3]:
Let $f_n$ be a sequence of measurable functions on $E$. Suppose there is a function $g$ that is integrable over $E$ and dominates $f_n$ on $E$ in the sense that $|f_n| \leq g$ on $E$ for all $n$. If $f_n \to f$ pointwise almost everywhere on $E$, then $f$ is integrable over $E$ and $\lim_{n \to \infty} \int_E f_n = \int_E f$

3. Dynamical system induced by $C_t$

In this section, we study the dynamical system induced by composition operator $C_t$ on various spaces like Banach space, Weighted Bergman space, Hilbert space, Weighted Hardy space, Weighted Lebesgue space, Dirichlet space, VMOA and the Little Bloch space.
Theorem 3.1. Let \( \{\phi_t\}_{t \geq 0} \) be a semigroup of analytic self maps on the unit disc \( D \) of \( C \). Let \( (X, \|\cdot\|_a) \) be a Banach space of analytic functions, analytic on \( D \) in which the polynomials \( P \) are dense. Let \( C_t : X \to X \) defined by \( C_t(f) = f \circ \phi_t \) be the bounded composition operator on \( X \) for \( t \geq 0 \). Then \( C_t \) induces a dynamical system on \( X \) iff \( \lim_{t \to 0} \|\phi^n_t - \psi_n\| = 0 \) where \( \psi_n(z) = z^n \)

Proof. We define \( \Delta(t, f) = f \circ \phi_t \), \( t \geq 0 \)

Clearly

(i) \( \Delta(o, f) = f \circ \phi_0 = f \)

(ii) \( \Delta(t, \Delta(s, f)) = \Delta(t, f \circ \phi_s) = f \circ \phi_s \circ \phi_t = f \circ \phi_{t+s} = \Delta(t+s, f) \)

(iii) We now show that \( \Delta(t, f) \) is continuous in both variables. Let \( f_n \to f \) then,

\[
|\Delta(t, f_n) - \Delta(t, f)| = |(f_n \circ \phi_t) - (f \circ \phi_t)| = |(f_n - f) \circ \phi_t|
\]

\( \to 0 \) as \( f_n \to f \)

To check the continuity in the first variable, let us show that \( \Delta(t, f) \to \Delta(0, f) \) as \( t \to 0 \).

\[
\|\Delta(t, f) - \Delta(0, f)\| = \|f \circ \phi_t - f\|
\]

\[
\leq \|f \circ \phi_t - p \circ \phi_t\| + \|p \circ \phi_t - p\| + \|p - f\|
\]

\[
= (\|C_t\| + 1)\|p - f\| + \|p \circ \phi_t - p\|
\]

Assume that there is a \( \delta > 0 \) such that \( \sup_{0 \leq t < \delta} \|C_t\| < \infty \). Since the polynomial \( p \) are dense in \( X \),

\[
\lim_{t \to 0} \|p \circ \phi_t - p\| = 0
\]

\( \Rightarrow \lim_{t \to 0} \|\phi^n_t - \psi_n\| = 0 \)
where $\psi_n(z) = z^n$, and so

$C_t$ induces a dynamical system on $X$.

**Theorem 3.2.** Let $\{\phi_t\}_{t \geq 0}$ be a semigroup of analytic self map on the unit disc $D$ of $C$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, every composition operator $C_t : X \to X$ defined by $C_t(f) = f \circ \phi_t$, $f \in A^p_\alpha$ induces a dynamical system on weighted Bergman space $A^p_\alpha$.

**Proof.** Since polynomials are dense in $A^p_\alpha$, By theorem 1, we have to show that

$$\lim_{t \to 0} \|\phi_t^n - \psi_n\|_{p,\alpha} = 0$$

where $\psi_n(z) = z^n$

$$\lim_{t \to 0} \|\phi_t^n - \psi_n\|_{p,\alpha} = \lim_{t \to 0} \frac{1}{\pi} \int_0^1 M_p(r, \phi_t^n - \psi_n)(1 - r^2)^{\alpha} r dr$$

Now, $M_p(r, \phi_t^n - \psi_n) = \int_0^{2\pi} |(\phi_t^n - \psi_n)(re^{i\theta})|^p d\theta$

$$= \int_0^{2\pi} |(\phi_t^n(re^{i\theta}) - r^n e^{in\theta})|^p d\theta$$

$$\leq 2^{p+1}\pi$$

Hence, $M_p(r, \phi_t^n - \psi_n)(1 - r^2)^{\alpha} r \leq 2^{p+1}\pi(1 - r^2)^{\alpha} r$ for each $t \geq 0$ and $r \in (0, 1)$.

Since $\alpha > -1$, the right hand side of the above inequality is integrable on $(0, 1)$. Using Lebesgue dominated convergence theorem we obtain

$$\lim_{t \to 0} \|\phi_t^n - \psi_n\|_{p,\alpha} = 0.$$

Hence $C_t$ induces a dynamical system on weighted Bergman space $A^p_\alpha$.

\[ \square \]

**Corollary 3.3.** If $1 \leq p < \infty$, then $C_t$ induces a dynamical system on a Banach space and if $p = 2$, then $C_t$ induces a dynamical system on a Hilbert space [1].

**Corollary 3.4.** Since the weighted Hardy space [6], weighted Lebesgue space [4], Dirichlet space, VMOA and the Little Bloch space [3] are the subspaces of the Banach space, $C_t$ induces a dynamical system on these spaces.
4. Some examples of dynamical systems on different spaces

In this section we discuss some examples of dynamical systems induced by the composition operator $C_t$ on different spaces like weighted Lebesgue space and weighted Hardy space.

**Example 4.1.** Let $\pi : R^+ \times L^p(T, w) \rightarrow L^p(T, w)$ be the function defined by $\pi(t, f) = e^{-t^2}f$ for $t \in R^+$ and $f \in L^p(T, w)$. Then $\pi$ is a dynamical system on $L^p(T, w)$.

*Proof.* (i) $\pi(0, f) = e^0 f = f$
(ii) $\pi(s + t, f) = e^{-(s^2 + t^2)}f$
(iii) $\|\pi(t, f) - \pi(s, f)\|$

$$= \left( \int_T \left| e^{-t^2}f - e^{-s^2}f \right|^p w(x)dx \right)^{\frac{1}{p}}$$

$$= \left( \int_T \left| e^{-t^2}f - e^{-s^2}f + e^{-s^2}f - e^{-s^2}f \right|^p w(x)dx \right)^{\frac{1}{p}}$$

$$= \left( \int_T \left| f(e^{-t^2} - 1) + e^{-s^2}(f - f) \right|^p w(x)dx \right)^{\frac{1}{p}}$$

$\rightarrow 0$ as $(t - s) \rightarrow 0$ and $(f - f) \rightarrow 0$

$\Rightarrow$ $\pi$ is continuous.
Hence $\pi$ is dynamical system on the weighted Lebesgue space $L^p(T, w)$. $\square$

**Example 4.2.** Let $D$ be the unit disc in the complex plane $C$ and let $h : D \rightarrow C$ be an analytic univalent function with $h(0) = 0$. Let $\pi : D \times H^p(\beta) \rightarrow H^p(\beta)$ be the function defined by $\pi(t, z) = h^{-1}(h(z) + ct)$, $z \in D$, $t \geq 0$. Then $\pi$ is a dynamical system on $H^p(\beta)$. 

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Proof. (i) \( \pi(0, z) = h^{-1}(h(z) + 0) = z \)
(ii) \( \pi(s + t, z) = h^{-1}(h(z) + c(s + t)) \) ———-(1)

\[
\begin{align*}
\pi(s, \pi(t, z)) &= \pi(s, h^{-1}(h(z) + ct)) \\
&= h^{-1}(h(h^{-1}(h(z) + ct)) + cs) \\
&= h^{-1}(h(z) + c(s + t)) ————(2)
\end{align*}
\]

From (1) and (2)
\( \pi(s + t, z) = \pi(s, \pi(t, z)) \)

(iii) \( \|\pi(t_n, z_n - \pi(t, z))\| \)
\[
= \sum_{n=0}^{\infty} (h^{-1}(h(z_n) + ct_n - h(z) - ct))^p \beta_p(n) \\
= \sum_{n=0}^{\infty} (h^{-1}(h(z_n) - h(z) + c(t_n - t)))^p \beta_p(n)
\]

Now the term \( h(z_n) - h(z) \to 0 \) as \( z_n - z \to 0 \) and \( c(t_n - t) \to 0 \) as \( t_n - t \to 0 \)
\( \Rightarrow \pi \) is continuous.

Hence \( \pi \) is dynamical system on \( H^p(\beta) \)

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