FIFTH ORDER PERIODIC BOUNDARY VALUE PROBLEM OF RANDOM DIFFERENTIAL EQUATION

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Abstract: In this paper, an existence of random solution is proved for a periodic boundary value problem of fifth order random differential equation through an algebraic random fixed point theorem of Dhage.

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1. DESCRIPTION OF THE PROBLEM

Let $R$ denote the real line and let $J = [0, 2\pi]$ be a closed and bounded interval in $R$. Let $C^4(J, R)$ denote the class of real-valued functions defined and continuously differentiable on $J$. Given a measurable space $(\Omega, A)$ and for a given measurable function $x : \Omega \rightarrow C^4(J, R)$, consider a fifth order periodic boundary value problem of ordinary random differential equations (in short PBVP)

\[ x^{(m)}(t, \omega) = f(t, x(t, \omega), \omega) \quad a.e. t \in J, \]
\[ x(0, \omega) = x(2\pi, \omega), \quad x'(0, \omega) = x'(2\pi, \omega), \]
\[ x''(0, \omega) = x''(2\pi, \omega), \quad x'''(0, \omega) = x'''(2\pi, \omega), \quad (1.1) \]

for all $\omega \in \Omega$, where $f : J \times R \times \Omega \rightarrow R$.

By a random solution of the random PBVP (1.1), we mean a measurable function $x : \Omega \rightarrow AC^4(J, R)$ that satisfies the equations in (1.1), where $AC^4(J, R)$ is the space of continuous real-valued functions, first derivative exists and absolutely continuous on $J$.

The random PBVP (1.1) is new to the theory of periodic boundary value problems of ordinary differential equations. When the random parameter $\omega$ is absent, the random PBVP (1.1) reduces to the classical PBVP of ordinary differential equations and the study of classical PBVP has been discussed in several papers by many authors for different aspects of the solutions. See for example, Lakshmikantham and Leela\textsuperscript{5}, Nieto \textsuperscript{6}, Yao \textsuperscript{8}, and the references therein. In this paper, we...
discuss the random PBVP (1.1) for existence of solution which generalize existence results of the classical PBVP.

2. AUXILIARY RESULTS

In this paper, we use the following random fixed point theorem for proving the main result. Theorem 2.1.(Dhage[3,4]) Let $U$ be a non-empty, open and bounded subset of the separable Banach space $E$ such that $0 \in U$ and let $Q : \Omega \times \overline{U} \to E$ be a compact and continuous random operator. Further suppose that there does not exist an $u \in \partial U$ such that $Q(\omega)x = \alpha x$ for all $\omega \in \Omega$, where $\alpha > 1$ and $\partial U$ is the boundary of $U$ in $E$. Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.

A corollary to above theorem in applicable form is

Corollary 2.1. Let $B_r(0)$ and $\overline{B}_r(0)$ be the open and closed balls centered at origin of radius $r$ in the separable Banach space $E$ and let $Q : \Omega \times \overline{B}_r(0) \to E$ be a compact and continuous random operator. Further suppose that there does not exist an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u = \alpha u$ for all $\omega \in \Omega$, where $\alpha > 1$. Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to \overline{B}_r(0)$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.

The following theorem is used in the study of nonlinear discontinuous random differential equations. We also quote this result.

Theorem 2.2.(Caratheodory) Let $Q : \Omega \times E \to E$ be a mapping such that $(\omega, \cdot)Q(\omega, \cdot)$ is measurable for all $\omega \in \Omega$ and $(\omega, \cdot)Q(\omega, \cdot)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, \cdot)Q(\omega, \cdot)$ is jointly measurable.

The following lemma appears in Nieto [6] and is useful in the study of periodic boundary value problems of ordinary differential equations.

Lemma 2.1. For any real number $m > 0$ and $\sigma \in L^1(J, R)$, $x$ is a solution to the differential equation

$$x'''(t) + m^2x(t) = \sigma(t) \quad a.e. \ t \in Jm,$$

$$x(0) = x(2\pi), x'(0) = x'(2\pi),$$

$$x'''(0) = x'''(2\pi), x'''(0) = x'''(2\pi),$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^{2\pi} G_m(t, s)\sigma(s)ds \quad (2.2)$$

where

$$G_m(t, s) = \frac{1}{2m(e^{m2\pi} - 1)} \left[ e^{m(t-s)} + e^{m(2\pi-t+s)} \right], \quad \text{if } 0 \leq s \leq t \leq 2\pi,$$

$$= \frac{1}{2m(e^{m2\pi} - 1)} \left[ e^{m(s-t)} + e^{m(2\pi-t+s)} \right], \quad \text{if } 0 \leq t < s \leq 2\pi \quad (2.3)$$

Notice that the Green’s function $G_m$ is continuous and nonnegative on $J \times J$ and the numbers
\[ \alpha = \min \left\{ \left| G_m(t, s) \right| : t, s \in [0, 2\pi] \right\} = \frac{e^{m \pi}}{m(e^{2m \pi} - 1)} \]

and
\[ \beta = \max \left\{ \left| G_m(t, s) \right| : t, s \in [0, 2\pi] \right\} = \frac{e^{2m \pi} + 1}{2m(e^{2m \pi} - 1)} \]

exist for all positive real number \( m \).

We need the following definitions.

Definition 2.1. A function \( f : J \times R \times \Omega \rightarrow R \) is called random Caratheodory if

(i) the map \((t, \omega) \rightarrow f(t, x, \omega)\) is jointly measurable for all \( x \in R \), and

(ii) the map \( x \rightarrow f(t, x, \omega) \) is continuous for all \( t \in J \) and \( \omega \in \Omega \).

Definition 2.2. A function \( f : J \times R \times \Omega \rightarrow R \) is called random \( L^1 \)-Carathéodory if

(iii) for each real number \( r > 0 \) there is a measurable and bounded function \( q_r : \Omega \rightarrow L^1(J, R) \) such that

\[ \left| f(t, x, \omega) \right| \leq q_r(t, \omega) \quad a.e. \ t \in J \quad \text{for all } \omega \in \Omega \text{ and } x \in R \text{ with } |x| \leq r. \]

3. EXISTENCE RESULT

For a given real number \( m > 0 \), consider the random PBVP,

\[ x''''(t, \omega) + m^2 x(t, \omega) = f_m(t, x(t, \omega), \omega) \quad a.e. t \in J, \]

\[ x(0, \omega) = x(2\pi, \omega), x'(0, \omega) = x'(2\pi, \omega), \]

\[ x''''(0, \omega) = x''''(2\pi, \omega), x''''(0, \omega) = x''''(2\pi, \omega), \] (3.1)

for all \( \omega \in \Omega \), where the function \( f_m : J \times R \times \Omega \rightarrow R \) is defined by

\[ f_m(t, x, \omega) = f(t, x, \omega) + m^2 x \]

Note: The random PBVP (1.1) is equivalent to the random PBVP (3.1) and therefore, a random solution to the PBVP (3.1) is the random solution to the PBVP (1.1) and vice versa.

The random PBVP (3.1) is equivalent to the random integral equation,

\[ x(t, \omega) = \int_0^{2\pi} G_m(s, t, \omega) f_m(s, x(s, \omega), \omega) ds \] (3.2)

for all \( t \in J \) and \( \omega \in \Omega \), where the function \( G_m(t, s) \) is defined by (2.3).

Consider the following two assumptions as

(\( A_1 \)) The function \( f_m \) is random Carathéodory on \( J \times R \times \Omega \).

(\( A_2 \)) There exists a measurable and bounded function \( \gamma : \Omega \rightarrow L^2(J, R) \) and a continuous and non-decreasing function \( \psi : R_+ \rightarrow (0, \infty) \) such that

\[ \left| f_m(t, x, \omega) \right| \leq \gamma(t, \omega) \psi(|x|) \quad a.e. t \in J \quad \text{for all } \omega \in \Omega \text{ and } x \in R. \]
4. MAIN EXISTENCE RESULT

Theorem 4.1. Assume that the hypotheses \((A_1)-(A_2)\) hold. Suppose that there exists a real number \(r > 0\) such that
\[
r > \frac{e^{2\pi r} + 1}{2m(e^{2\pi r} - 1)} \|\psi(\omega)\|_L^2 \psi(r) \tag{4.1}
\]
for all \(\omega \in \Omega\). Then the random PBVP (1.1) has a random solution defined on \(J\).

Proof: Set \(E = C(J, R)\) and define a mapping \(Q: \Omega \times E \to E\) by
\[
Q(\omega)x(t) = \int_0^{2\pi} G_m(t, s) f_m(s, x(s, \omega), \omega) ds
\]
for all \(t \in J\) and \(\omega \in \Omega\). Since the map \(t \to G_m(t, s)\) is continuous on \(J\), \(Q(\omega)\) defines a mapping \(Q: \Omega \times E \to E\). Define a closed ball \(B_r(0)\) in \(E\) centered at origin \(0\) of radius \(r\), where the real number \(r\) satisfies the inequality (3.3). We show that \(Q\) satisfies all the conditions of Corollary 2.1 on \(B_r(0)\).

First, we show that \(Q\) is a random operator on \(B_r(0)\). Since \(f_m(t, x, \omega)\) is random Caratheodory, the map \(\omega \to f_m(t, x, \omega)\) is measurable in view of Theorem 2.2. Similarly, the product \(G_m(t, s) f_m(s, x(s, \omega), \omega)\) of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map
\[
\omega \to \int_0^{2\pi} G_m(t, s) f_m(s, x(s, \omega), \omega) ds = Q(\omega)x(t)
\]
is measurable. As a result, \(Q\) is a random operator on \(\Omega \times B_r(0)\) into \(E\).

Next, we show that the random operator \(Q(\omega)\) is continuous on \(B_r(0)\). Let \(\{x_n\}\) be a sequence of points in \(B_r(0)\) converging to the point \(x\) in \(B_r(0)\). Then, it is enough to prove that \(\lim_{n \to \infty} Q(\omega)x_n(t) = Q(\omega)x(t)\) for all \(t \in J\) and \(\omega \in \Omega\). By dominated convergence theorem, we obtain,
\[
\lim_{n \to \infty} Q(\omega)x_n(t) = \lim_{n \to \infty} \int_0^{2\pi} G_m(t, s) f_m(s, x_n(s, \omega), \omega) ds
\]
\[
= \lim_{n \to \infty} \int_0^{2\pi} G_m(t, s) \lim_{n \to \infty} [f_m(s, x_n(s, \omega), \omega)] ds
\]
\[
= \int_0^{2\pi} G_m(t, s) [f_m(s, x(s, \omega), \omega)] ds = Q(\omega)x(t)
\]
for all \(t \in J\) and \(\omega \in \Omega\). This shows that \(Q(\omega)\) is a continuous random operator on \(B_r(0)\).
Now, we show that $Q(\omega)$ is a compact random operator on $\bar{B}_r(0)$. To finish, it is enough to prove that $Q(\omega)(\bar{B}_r(0))$ is uniformly bounded and equi-continuous set in $E$ for each $\omega \in \Omega$. Since the map $\omega \to \gamma(t, \omega)$ is bounded and $L^2(J, R) \subset L^2(J, R)$ by hypothesis (A), there is constant $c$ such that $\|\gamma(\omega)\|_2 \leq c$ for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. Then for any $x : \Omega \to \bar{B}_r(0)$, one has

$$
\left|Q(\omega)x(t)\right| \leq \int_0^{2\pi} G_m(t, s)\left|f_m(s, x(s, \omega), \omega)\right| \, ds
$$

$$
\leq e^{2m\pi} + 1 \left(\int_0^{2\pi} \gamma(s, \omega) \, ds\right) \psi(r) \leq K_1
$$

for all $t \in J$, where $K_1 = \frac{e^{2m\pi} + 1}{2m(e^{2m\pi} - 1)} c \psi(r)$. This shows that $Q(\omega)(\bar{B}_r(0))$ is a uniformly bounded subset of $E$ for each $\omega \in \Omega$.

Next, we show that $Q(\omega)(\bar{B}_r(0))$ is an equi-continuous set in $E$. Let $x \in \bar{B}_r(0)$ be arbitrary. Then for any $t_1, t_2 \in J$, one has

$$
\left|Q(\omega)x(t_1) - Q(\omega)x(t_2)\right| \leq \int_0^{2\pi} \left|G_m(t_1, s) - G_m(t_2, s)\right| \left|f_m(s, x(s, \omega), \omega)\right| \, ds
$$

$$
\leq \int_0^{2\pi} \left|G_m(t_1, s) - G_m(t_2, s)\right|^2 \, ds \frac{1}{2\pi} \left(\int_0^{2\pi} \gamma(s, \omega) \, ds\right)^2 \psi(r)
$$

Hence for all $t_1, t_2 \in J$, $\left|Q(\omega)x(t_1) - Q(\omega)x(t_2)\right| \to 0$ as $t_1 \to t_2$, uniformly for all $x \in \bar{B}_r(0)$. Therefore, $Q(\omega)(\bar{B}_r(0))$ is an equi-continuous set in $E$. As $Q(\omega)(\bar{B}_r(0))$ is uniformly bounded and equi-continuous, it is compact by Arzel-Ascoli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on $\bar{B}_r(0)$.

Finally, we prove that there does not exist an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u(t) = \alpha u(t, \omega)$ for all $t \in J$ and $\omega \in \Omega$, where $\alpha > 1$. Suppose not. Then there exists an element $u \in E$ satisfying $Q(\omega)u(t) = \alpha u(t, \omega)$ for some $\omega \in \Omega$. Let $\lambda = \frac{1}{\alpha}$. Then $\lambda < 1$ and

$$
\lambda Q(\omega)u(t) = u(t, \omega)
$$

for some $\omega \in \Omega$. Now for this $\omega \in \Omega$, one has

$$
\left\|u(t, \omega)\right\| \leq \lambda \left|Q(\omega)u(t)\right|
$$

$$
\leq \int_0^{2\pi} G_m(t, s)\left|f_m(s, u(s, \omega), \omega)\right| \, ds
$$
\[
\begin{align*}
\leq & \frac{e^{2m\sigma} + 1}{2m(e^{2m\sigma} - 1)} \int_{0}^{2\pi} \gamma(s, \omega)\psi(\|u(\omega)\|)ds \\
\leq & \frac{e^{2m\sigma} + 1}{2m(e^{2m\sigma} - 1)} \|\gamma(\omega)\|_{L} \psi(\|u(\omega)\|)
\end{align*}
\]
for all \( t \in J \). Taking supremum over \( t \) in the above inequality yields,
\[
\|u(\omega)\| \leq \frac{e^{2m\sigma} + 1}{2m(e^{2m\sigma} - 1)} \|\gamma(\omega)\|_{L} \psi(\|u(\omega)\|)
\]
or
\[
r \leq \frac{e^{2m\sigma} + 1}{2m(e^{2m\sigma} - 1)} \|\gamma(\omega)\|_{L} \psi(r)
\]
for some \( \omega \in \Omega \). This contradicts to the condition (3.3).

Thus, all the conditions of Corollary 2.1 are satisfied. Hence the random equation \( Q(\omega)x(t) = x(t, \omega) \) has a random solution in \( B_{r}(\Omega) \), i.e., there is a measurable function \( \xi : \Omega \rightarrow B_{r}(\Omega) \) such that \( Q(\omega)\xi(t) = \xi(t, \omega) \) for all \( t \in J \) and \( \omega \in \Omega \). As a result, the random PBVP (1.1) has a random solution defined on \( J \). This completes the proof.

REFERENCES


