On Intuitionistic Fuzzy Rough Sets

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Abstract

The extension of rough set model is an important research direction in rough set theory. The aim of this paper is to derive a new extension of classical notion of set theory. We initiate the concept of intuitionistic fuzzy rough topology in an approximation space based on an intuitionistic fuzzy equivalence relation. Furthermore some properties are characterized for this topology.

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1. Introduction

Rough set theory, proposed by Pawlak[11] is a new mathematical tool that supports uncertainty reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields. The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximation spaces are induced. Using this approximation knowledge hidden information may be revealed and expressed in the form of decision rules. A key notion in Pawlak[11] rough set model is an equivalence relation. Atanassov[1] presented intuitionistic fuzzy sets in 1986 which is very effective to deal with vagueness. As a generalization of fuzzy set the concept of intuitionistic fuzzy set has played an important role in analysis of uncertainty of data. Various notions of intuitionistic fuzzy rough set were explored to extend rough set theory in the intuitionistic fuzzy environment. In this paper we introduce intuitionistic fuzzy rough set in topological space and provide some of its basic properties. We utilize the notion of indiscernibility from rough set theory coupled with the
idea of membership and non membership values from IF set theory.

2. Preliminaries

Definition 2.1[11]: Let $U$ be a non empty finite set of objects called the universe and $R$ be an equivalence relation on $U$ named as indiscemibity relation. The pair $(U, R)$ is called the approximation space.

Definition 2.2[11]: Let $X$ be a subset of $U$

(i) The lower approximation of $X$ with respect to $R$ is the set of all objects which can be for certain classified as $X$ with respect to $R$ and it is denoted by

$$R(X) = \{ R(x) : R(x) \subseteq X \}$$

where $R(x)$ denotes the equivalence class determined by $x$.

(ii) The upper approximation of $X$ with respect to $R$ is the set of all objects which can be possibly classified in $X$ with respect to $R$ and it is denoted by

$$\overline{R}(X) = \{ R(x) : R(x) \cap X \neq \phi \}.$$ 

Proposition 2.3[16]: If $(U, R)$ be an approximation space and $X$ and $Y$ subsets of $U$ then

1. $\overline{R}(X) \subseteq X \subseteq R(X)$.
2. $R(\phi) = \overline{R}(\phi) = \phi, R(U) = \overline{R}(U) = U$.
3. $R(X \cup Y) = R(X) \cup R(Y)$.
4. $R(X \cup Y) \supseteq R(X) \cup R(Y)$.
5. $R(X \cap Y) = R(X) \cap R(Y)$.
6. $R(X \cap Y) \subseteq R(X) \cap R(Y)$.
7. $R(X) \subseteq R(Y)$ and $\overline{R}(X) \subseteq \overline{R}(Y)$ whenever $X \subseteq Y$.
8. $R(X^c) = [\overline{R}(X)]^c$ and $\overline{R}(X^c) = [R(X)]^c$.
9. $R(R(X)) = \overline{R}(R(X)) = R(X)$.
10. $\overline{R}(\overline{R}(X)) = \overline{R}(\overline{R}(X)) = \overline{R}(X)$.

**Definition 2.4[1]:** An intuitionistic fuzzy set (IFS in short) $A$ in $X$ is an object having
the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \}$ where the function $\mu : X \rightarrow [0,1]$ and $\nu : X \rightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\nu_A(x)$) of each element $x \in X$ to the set $A$, respectively and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by $\text{IFS}(X)$ the set of all intuitionistic fuzzy set in $X$.

**Definition 2.5[1]:** Let $A$ and $B$ be IFS’s of the form $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \}$ and $B = \{\langle x, \mu_B(x), \nu_B(x) \rangle / x \in X \}$. Then

1. $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $A^c = \{\langle x, \nu_A(x), \mu_A(x) \rangle / x \in X \}$.
4. $A \cap B = \{\langle x, x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x) \rangle / x \in X \}$.
5. $A \cup B = \{\langle x, x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x) \rangle / x \in X \}$.

For the sake of simplicity we use the notion $A = \langle x, \mu_A, \nu_A \rangle$ instead of $A = \{\langle x, \mu_A(x), \nu_A(x) \rangle / x \in X \}$. The intuitionistic fuzzy set $0 \sim = \{\langle x, 0 \sim, 1 \sim \rangle / x \in X \}$ and $1 \sim = \{\langle x, 1 \sim, 0 \sim \rangle / x \in X \}$ are respectively the empty set and the whole set of $X$.

**Definition 2.6[7]:** Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$. The product $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by

$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))\forall(x_1, x_2) \in X_1 \times X_2$$

**Definition 2.7[7]:** Let $f : X \rightarrow Y$ be a function. The graph $g : X \rightarrow X \times Y$ of $f$ is defined by

$$g(x) = (x, f(x))\forall x \in X.$$ 

**Definition 2.8[7]:** Let $(X, \Psi)$ and $(Y, \Phi)$ be IFTS’s. The intuitionistic fuzzy product space(IFPTS, for short) of $(X, \Psi)$ and $(Y, \Phi)$ is the cartesian product $X \times Y$ of IFS’s $X$ and $Y$ together with the IFT $\xi$ of $X \times Y$ which is generated by the family $\{P_i^{-1}(A_i), P_j^{-1}(B_j) : A_i \in \Psi, B_j \in \Phi \}$ and $P_i, P_j$ are projections of $X \times Y$ onto $X$ and $Y$, respectively.
(i.e. the family $P_1^{-1}(A_i), P_2^{-1}(B_j) : A_i \in \Psi, B_j \in \Phi$ is a sub base for IFT $\xi$ of $X \times Y$.

**Definition 2.9[16]:** Let $U$ be a non-empty universe of discourse and $R$ an intuitionistic fuzzy relation on $U$, then the pair $(U,R)$ is called an intuitionistic fuzzy approximation space. For $A \in IF(U)$ the upper and lower approximation of $A$ with respect to $(U,R)$ are denoted by $\overline{R}(A)$ and $\overline{R}(A)$ are two IF sets and are, respectively, defined as follows:

$$\overline{R}(A) = \{ (x, \mu_{\overline{R}(A)}(x), \nu_{\overline{R}(A)}(x)) / x \in U \}$$

$$\underline{R}(A) = \{ (x, \mu_{\underline{R}(A)}(x), \nu_{\underline{R}(A)}(x)) / x \in U \}$$

where

$$\mu_{\overline{R}(A)}(x) = \bigwedge_{y \in U} \mu_{R}(x,y) \land \mu_{A}(y),$$

$$\nu_{\overline{R}(A)}(x) = \bigwedge_{y \in U} \nu_{R}(x,y) \lor \nu_{A}(y),$$

$$\mu_{\underline{R}(A)}(x) = \bigvee_{y \in U} \mu_{R}(x,y) \lor \mu_{A}(y),$$

$$\nu_{\underline{R}(A)}(x) = \bigvee_{y \in U} \nu_{R}(x,y) \land \nu_{A}(y).$$

The pair $(R(A), \overline{R}(A))$ is called the IF rough set of $A$ w.r.t $(U, R)$ and $\overline{R}, R : IF(U) \rightarrow IF(U)$ are referred to as the upper and lower IF rough approximation operators, respectively.

**Definition 2.10[16]:** An intuitionistic fuzzy binary relation on $U$ is an intuitionistic fuzzy subset of $U \times U$, namely, $R$ is given by

$$R = \{ ((x,y), \mu_{R}(x,y), \nu_{R}(x,y)) / x,y \in U \}$$

$$\mu_{R} : U \times U \rightarrow [0,1], \nu_{R} : U \times U \rightarrow [0,1]$$

satisfy $0 \leq \mu_{R}(x,y)+\nu_{R}(x,y) \leq 1$ for all $(x,y) \in U \times U$. IFR$(U \times U)$ will be used to denote the family of all intuitionistic fuzzy relation on $U$.

For an intuitionistic fuzzy relation $R \in IFR(U \times U)$ we say that $R$ is

1. reflexive if $\mu_{R}(x,x) = 1$ and $\nu_{R}(x,x) = 0$ for all $x \in U$.

2. symmetric if $\mu_{R}(x,y) = \mu_{R}(y,x)$ and $\nu_{R}(x,y) = \nu_{R}(y,x)$ for all $(x,y) \in U \times U$.

3. transitive if for every $(x,z) \in U \times U$

$$\mu_{R}(x,z) \geq \bigwedge_{y \in U} [\mu_{R}(x,y) \land \mu_{R}(y,z)]$$

$$\nu_{R}(x,z) \leq \bigvee_{y \in U} [\nu_{R}(x,y) \lor \nu_{R}(y,z)]$$

**3. Intuitionistic fuzzy rough sets**

**Definition 3.1:** Let $U$ be a non-empty universe and $R$ an intuitionistic fuzzy equivalence relation on $U$, the pair $(U,R)$ is called an intuitionistic fuzzy approximation space. For $A \in (U, R)$ the upper and lower approximation of $A$ with respect to $(U,R)$ are denoted by $\overline{R}(A)$ and $\overline{R}(A)$. Then an intuitionistic fuzzy rough set (IFRS) in an approximation space is
defined as $\mathcal{R}(A) = (\overline{R}(A), \overline{\mathcal{R}}(A))$.

where

\[
\begin{align*}
\overline{R}(A) &= \{ \langle u, \mu_{\overline{R}(A)}(u), \nu_{\overline{R}(A)}(u) \rangle / u \in U \} \\
\overline{\mathcal{R}}(A) &= \{ \langle u, \mu_{\overline{\mathcal{R}}(A)}(u), \nu_{\overline{\mathcal{R}}(A)}(u) \rangle / u \in U \}
\end{align*}
\]

with

\[
\begin{align*}
\mu_{\overline{R}(A)} &= \vee_{v \in U} [\mu_{R}(A)(u, v) \land \mu_A(v)] \\
\nu_{\overline{R}(A)} &= \wedge_{v \in U} [\nu_{R}(A)(u, v) \lor \nu_A(v)] \\
\mu_{\overline{\mathcal{R}}(A)} &= \wedge_{v \in U} [\mu_{\overline{R}(A)}(u, v) \lor \mu_A(v)] \\
\nu_{\overline{\mathcal{R}}(A)} &= \vee_{v \in U} [\nu_{\overline{R}(A)}(u, v) \land \nu_A(v)]
\end{align*}
\]

also

\[
\begin{align*}
\mu_{\overline{R}(A)} : R(A) &\to [0, 1] \quad \text{and} \quad \nu_{\overline{R}(A)} : R(A) &\to [0, 1] \\
\mu_{\overline{\mathcal{R}}(A)} : R(A) &\to [0, 1] \quad \text{and} \quad \nu_{\overline{\mathcal{R}}(A)} : R(A) &\to [0, 1]
\end{align*}
\]

**Proposition 3.2** : Let $(U, R)$ be an intuitionistic fuzzy approximation space, $A$ and $B$ subsets of $U$ then

1. $R(A) \subseteq A \subseteq \overline{R}(A)$.
2. $R(0 \sim) = \overline{R}(0 \sim) = 0 \sim$.
3. $R(1 \sim) = \overline{R}(1 \sim) = 1 \sim$.
4. $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$.
5. $R(A \cup B) \supseteq R(A) \cup R(B)$.
6. $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$.
7. $\overline{R}(A \cap B) = \overline{R}(A) \cap \overline{R}(B)$.
8. If $A \subseteq B$ then $R(A) \subseteq R(B)$ and $\overline{R}(A) \subseteq \overline{R}(B)$.
9. $R(A^c) = [\overline{R}(A)]^c$ and $\overline{R}(A^c) = [\overline{R}(A)]^c$. 

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10. \( R(R(A)) = R(A) \).

11. \( \overline{R}(R(A)) = R(A) \).

12. \( R(A) + R(B) = (R(A) + R(B), \overline{R}(A) + \overline{R}(B)) \)
   \[
   \overline{R}(A) + R(B) = \left\{ u, \mu_{R(A)}(u) + \mu_{R(B)}(u) - \mu_{R(A)}(u) \cdot \mu_{R(B)}(u), \mu_{R(A)}(u) \cdot \mu_{R(B)}(u) \right\}
   \]
   \[
   \overline{R}(A) + \overline{R}(B) = \left\{ u, \mu_{\overline{R}(A)}(u) + \mu_{\overline{R}(B)}(u) - \mu_{\overline{R}(A)}(u) \cdot \mu_{\overline{R}(B)}(u), \mu_{\overline{R}(A)}(u) \cdot \mu_{\overline{R}(B)}(u) \right\}
   \]

Definition 3.3 : Let \( U \) and \( V \) be two non empty universal sets.

(i) If \( R(B) = (R(B), \overline{R}(B)) \) is an intuitionistic fuzzy rough set in \( V \), where \( R(B) = \langle v, \mu_{R(B)}(v), \nu_{R(B)}(v) \rangle \), \( \overline{R}(B) = \langle v, \mu_{\overline{R}(B)}(v), \nu_{\overline{R}(B)}(v) \rangle \), then the preimage of \( R(B) \) under \( f \), denoted by \( f^{-1}(R(B)) \) is the intuitionistic fuzzy rough set in \( V \) defined by

\[
\overline{R}(R(B)) = (f^{-1}(R(B)), f^{-1}(\overline{R}(B)))
\]

where

\[
f^{-1}(R(B)) = \langle u, f^{-1}(\mu_{R(B)}(u)), f^{-1}(\nu_{R(B)}(u)) \rangle
\]

\[
f^{-1}(\overline{R}(B)) = \langle u, f^{-1}(\mu_{\overline{R}(B)}(u)), f^{-1}(\nu_{\overline{R}(B)}(u)) \rangle
\]

(ii) If \( R(A) = (R(A), \overline{R}(A)) \) is an intuitionistic fuzzy rough set in \( V \), then the image of \( R(A) \) under \( f \) is denoted by \( f(R(A)) \) is the intuitionistic fuzzy rough set in \( V \) defined by

\[
f(R(A)) = (f(R(A)), f(\overline{R}(A)))
\]

where

\[
f(R(A)) = \langle v, f(\mu_{R(A)}(v)), 1 - f(1 - \nu_{R(A)}(v)) \rangle
\]

\[
f(\overline{R}(A)) = \langle v, f(\mu_{\overline{R}(A)}(v)), 1 - f(1 - \nu_{\overline{R}(A)}(v)) \rangle
\]

and

\[
f(\mu_{R(A)}(v)) = \begin{cases} 
\sup_{u \in f^{-1}(v)} \mu_{R(A)}(u) & \text{if } f^{-1}(v) \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]
Let $\mathcal{R}$ and $\mathcal{F}$ be intuitionistic fuzzy rough set in $V$ and $R$. For the sake of simplicity let us use the symbol $f_{-}(\nu_{\mathcal{R}(A)})$ for $1 - f(1 - \nu_{\mathcal{R}(A)})$ and $f_{-}(\nu_{\mathcal{R}(A)})$ for $1 - f(1 - \nu_{\mathcal{R}(A)})$. Now we shall list the properties of images and preimages some of which we shall frequently use.

**Corollary 3.4** Let $\mathcal{R}(A_i)(i \in J)$ be intuitionistic fuzzy rough set in $U, \mathcal{R}(B_j)(j \in K)$ intuitionistic fuzzy rough set in $V$ and $f : U \to V$ be a function. Then

(a) $\mathcal{R}(A_1) \subseteq \mathcal{R}(A_2) \Rightarrow f(\mathcal{R}(A_1)) \subseteq f(\mathcal{R}(A_2))$.

(b) $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2) \Rightarrow f^{-1}(\mathcal{R}(B_1)) \subseteq f^{-1}(\mathcal{R}(B_2))$.

(c) $\mathcal{R}(A) \subseteq f^{-1}(f(\mathcal{R}(A)))$. [If $f$ is injective then $\mathcal{R}(A) = f^{-1}(f(\mathcal{R}(A)))$].

(d) $f(f^{-1}(\mathcal{R}(B)) \subseteq \mathcal{R}(B)$. [If $f$ is surjective then $f(f^{-1}(\mathcal{R}(B)) = \mathcal{R}(B)$].

(e) $f^{-1}(\bigcup \mathcal{R}(B_j)) = \bigcup f^{-1}(\mathcal{R}(B_j))$.

(f) $f^{-1}(\bigcap \mathcal{R}(B_j)) = f^{-1}(\mathcal{R}(B_j))$.

(g) $f(\bigcup \mathcal{R}(A_i)) = f(\bigcup \mathcal{R}(A_i))$.

(h) $f(\bigcap \mathcal{R}(A_i)) \subseteq \bigcap f(\mathcal{R}(A_i))$. [If $f$ is injective, then $f(\bigcap \mathcal{R}(A_i) = \bigcap f(\mathcal{R}(A_i))]$

(i) $f^{-1}(1 \sim) = 1 \sim$.

(j) $f^{-1}(0 \sim) = 0 \sim$.

(k) $f(1 \sim) = 1 \sim$, if $f$ is surjective.

(l) $f(0 \sim) = 0 \sim$.

(m) $(f(\mathcal{R}(A))^c \subseteq f(\mathcal{R}(A))^c$, if $f$ is surjective.

(n) $f^{-1}(\mathcal{R}(B))^c = (f^{-1}(\mathcal{R}(B))^c$.

**Proof:** Let $\mathcal{R}(B_j) = (\mathcal{R}(B_j), \mathcal{R}(B_j))$ where $\mathcal{R}(B_j) = (v, \mu_{\mathcal{R}(B_j)}, \nu_{\mathcal{R}(B_j)}), \mathcal{R}(B_j) = (v, \mu_{\mathcal{R}(B_j)}, \nu_{\mathcal{R}(B_j)}))$ and $\mathcal{R}(A_i) = (\mathcal{R}(A_i), \mathcal{R}(A_i))$ where
\( \overline{R}(A_i) = \langle u, \mu_{R(B)}, \nu_{R(B)} \rangle, \overline{R}(A_i) = \langle u, \mu_{\overline{R}(A_i)}, \nu_{\overline{R}(A_i)} \rangle \)

then 

(a) \( f(\overline{R}(A_1)) \subseteq f(\overline{R}(A_2)) \) (i.e. \( f(\overline{R}(A_1)) \subseteq f(\overline{R}(A_2)) \) and \( f(\overline{R}(A_1)) \subseteq f(\overline{R}(A_2)) \). Since \( \mu_{\overline{R}(A_i)} \leq \mu_{\overline{R}(A_2)} \) and \( \nu_{\overline{R}(A_i)} \geq \nu_{\overline{R}(A_2)} \) we obtain 

\( f(\mu_{\overline{R}(A_1)}) \leq f(\mu_{\overline{R}(A_2)}) \) and \( 1 - \nu_{\overline{R}(A_1)} \leq 1 - \nu_{\overline{R}(A_2)} \) implies \( f(1 - \nu_{\overline{R}(A_i)}) \leq f(1 - \nu_{\overline{R}(A_2)}) \) \( \Rightarrow 1 - f(1 - \nu_{\overline{R}(A_1)}) \geq 1 - f(1 - \nu_{\overline{R}(A_2)}) \Rightarrow f(\nu_{\overline{R}(A_1)}) \leq f(\nu_{\overline{R}(A_2)}) \). Thus \( f(\overline{R}(A_1)) \subseteq f(\overline{R}(A_2)) \).

In a similar manner we obtain \( f(\overline{R}(A_1)) \subseteq f(\overline{R}(A_2)) \).

(b) The proof is similar to (a).

(c) \( f^{-1}(f(\overline{R}(A))) = f^{-1}(f(\overline{R}(A), \overline{R}(A))) = (f^{-1}(f(\overline{R}(A))), f^{-1}(f(\overline{R}(A)))) \)

\( f^{-1}(f(\overline{R}(A))) = f^{-1}(v, f(\mu_{\overline{R}(B)}), f_{\nu_{\overline{R}(B)}}) \)

\( = \langle u, f^{-1}(f(\mu_{\overline{R}(B)})), f_{\nu_{\overline{R}(B)}} \rangle \supseteq \langle u, \mu_{\overline{R}(A)}, \nu_{\overline{R}(A)} \rangle = \overline{R}(A) \)

[Note \( f^{-1}(f(\mu_{\overline{R}(B)})) \geq \mu_{\overline{R}(A)} \) and \( f^{-1}(f(1 - \nu_{\overline{R}(B)})) \leq 1 - (1 - \nu_{\overline{R}(A)}) \)

\( = \nu_{\overline{R}(A)} \).

Similarly \( f^{-1}(f(\overline{R}(A))) \supseteq \langle u, \mu_{\overline{R}(A)}, \nu_{\overline{R}(A)} \rangle = \overline{R}(A) \).

Hence \( \overline{R}(A) \subseteq f^{-1}(f(\overline{R}(A))) \).

(d) \( f(f^{-1}(f(\overline{R}(B)))) = f(f^{-1}(f(\overline{R}(B)), f(\overline{R}(B)))) = (f^{-1}(f(\overline{R}(B))), f^{-1}(f(\overline{R}(B)))) \)

\( f(f^{-1}(f(\overline{R}(B))) = f^{-1}(v, f(\mu_{\overline{R}(B)}), f_{\nu_{\overline{R}(B)}}) \)

\( = f(\langle u, f^{-1}(f(\mu_{\overline{R}(B)})), f_{\nu_{\overline{R}(B)}} \rangle) = \langle u, f^{-1}(f(\mu_{\overline{R}(B)})), f_{\nu_{\overline{R}(B)}} \rangle \)

\( \subseteq \langle u, \mu_{\overline{R}(B)}, \nu_{\overline{R}(B)} \rangle = \overline{R}(B) \)

[Note \( f(f^{-1}(f(\mu_{\overline{R}(B)}))) \leq \mu_{\overline{R}(B)} \) and \( f_{\nu_{\overline{R}(B)}} \geq \nu_{\overline{R}(B)} \).

Similarly \( f(f^{-1}(f(\overline{R}(B)))) \supseteq \overline{R}(B) \). Hence \( \overline{R}(A) \subseteq f^{-1}(f(\overline{R}(A))). \)

(e) \( f^{-1}(f(\overline{R}(B_1), \overline{R}(B_2))) = f^{-1}(f(\overline{R}(B_1), \overline{R}(B_2))) \)

\( = \langle u, f^{-1}(\langle \mu_{\overline{R}(B_1)}, \nu_{\overline{R}(B_1)} \rangle), f_{\nu_{\overline{R}(B_1)}} \rangle = \langle u, f^{-1}(\langle \mu_{\overline{R}(B_1)}, \nu_{\overline{R}(B_1)} \rangle), f_{\nu_{\overline{R}(B_1)}} \rangle \)

\( = \langle u, \mu_{\overline{R}(B_1)}, f_{\nu_{\overline{R}(B_1)}} \rangle \cup f^{-1}(f(\overline{R}(B_2))) \)

Similarly \( f^{-1}(f(\overline{R}(B_1))) = f^{-1}(f(\overline{R}(B_2))). \)

(f) Similar to (e).

(g) \( f(\bigcup (R(A_i))) = (f(\bigcup (R(A_i))), f(\bigcup (R(A_i))) \)

\( f(\bigcup (R(A_i))) = f(\langle u, \mu_{R(A_i)}, f_{\nu_{R(A_i)}} \rangle) = \langle v, f(\bigcup (R(A_i))), f_{\nu_{R(A_i)}} \rangle \)

\( = \langle v, f(\mu_{R(A_i)}), f_{\nu_{R(A_i)}} \rangle = \bigcup f(\mu_{R(A_i)}), f_{\nu_{R(A_i)}} \rangle \)

\( = \bigcup f(\overline{R}(A_i)) \)

[Note: \( f(\bigcup (R(A_i))) = f(\mu_{R(A_i)}), f_{\nu_{R(A_i)}} \rangle = f_{\nu_{R(A_i)}}, f_{\nu_{R(A_i)}} \rangle \)

Similarly \( f(\bigcup (\overline{R}(A_i)) = \bigcup f(\overline{R}(A_i)). \)

(h) \( f(\bigcap (R(A_i))) = (f(\bigcap (R(A_i))), f(\bigcap (R(A_i))) \)

\( f(\bigcap (R(A_i))) = f(\langle u, \mu_{R(A_i)}, f_{\nu_{R(A_i)}} \rangle) = \langle v, f(\bigcap (R(A_i))), f_{\nu_{R(A_i)}} \rangle \)

\( \subseteq \langle v, \mu_{R(A_i)}, f_{\nu_{R(A_i)}} \rangle \cup f_{\nu_{R(A_i)}} \rangle = \bigcap f(\overline{R}(A_i)). \)
with intuitionistic fuzzy rough set of $U$ and $V$ respectively. Then

$$f(\bigcap \mathcal{R}(A_i)) \subseteq \bigcap f(\mathcal{R}(A_i)).$$

Similarly we can prove that $f(\bigcap \mathcal{R}(A_i)) \subseteq \bigcap f(\mathcal{R}(A_i)).$

(i),(j),(k),(l) are obvious.

(m) Since $f(\mathcal{R}(A)) = (f(\mathcal{R}(A))^c, (f(\mathcal{R}(A)))^c).

(f(\mathcal{R}(A)))^c = f(\langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle) = \langle v, f(\nu_{\mathcal{R}(A)}(u)), f_{-}(\mu_{\mathcal{R}(A)}(u)) \rangle.

Similarly \((f(\mathcal{R}(A)))^c = \langle v, f(\nu_{\mathcal{R}(A)}(u)), f_{-}(\mu_{\mathcal{R}(A)}(u)) \rangle\) and

\((f(\mathcal{R}(A)))^c = (f(\mathcal{R}(A)))^c, (f(\mathcal{R}(A)))^c).\)

Thus we obtain the required result immediately from the fact that \(f\) is surjective.

(n) It is similar to (m).

**Definition 3.5:** Let $U$ and $V$ be non empty universal set and $\mathcal{R}(A) = (\mathcal{R}(A), \mathcal{R}(A))$ where

$$\mathcal{R}(A) = \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle \quad \mathcal{R}(A) = \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle$$

and $\mathcal{R}(B) = (\mathcal{R}(B), \mathcal{R}(B))$ where

$$\mathcal{R}(B) = \langle v, \mu_{\mathcal{R}(B)}(v), \nu_{\mathcal{R}(B)}(v) \rangle \quad \mathcal{R}(B) = \langle v, \mu_{\mathcal{R}(B)}(v), \nu_{\mathcal{R}(B)}(v) \rangle.$$

be intuitionistic fuzzy rough set of $U$ and $V$ respectively. Then $\mathcal{R}(A) \times \mathcal{R}(B)$ is an intuitionistic fuzzy rough set of $U \times V$ defined by $(\mathcal{R}(A) \times \mathcal{R}(B))(u,v) = (\mathcal{R}(A) \times \mathcal{R}(B), (\mathcal{R}(A) \times \mathcal{R}(B))(u,v)$ where

\((\mathcal{R}(A) \times \mathcal{R}(B))(u,v) = \langle (u,v), \min(\mu_{\mathcal{R}(A)}(u), \mu_{\mathcal{R}(B)}(v)), \max(\nu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(B)}(v)) \rangle \)

\((\mathcal{R}(A) \times \mathcal{R}(B))(u,v) = \langle (u,v), \min(\mu_{\mathcal{R}(A)}(u), \mu_{\mathcal{R}(B)}(v)), \max(\nu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(B)}(v)) \rangle \)

with

$$\min(\mu_{\mathcal{R}(A)}(u), \mu_{\mathcal{R}(B)}(v)) = (\mu_{\mathcal{R}(A)} \times \mu_{\mathcal{R}(B)})(u,v)$$

$$\max(\nu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(B)}(v)) = (\nu_{\mathcal{R}(A)} \times \nu_{\mathcal{R}(B)})(u,v)$$

Note that

$$1 - (\mathcal{R}(A) \times \mathcal{R}(B))(u,v) = \langle (u,v), \max(\nu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(B)}(v)), \min(\mu_{\mathcal{R}(A)}(u), \mu_{\mathcal{R}(B)}(v)) \rangle$$

**Lemma 3.6:** If $A$ is an intuitionistic fuzzy rough set of $U$ and $B$ is an intuitionistic fuzzy rough set of $V$ then
(i) \((\mathcal{R}(A) \times 1 \sim) \cap (1 \sim \times \mathcal{R}(B)) = \mathcal{R}(A) \times \mathcal{R}(B)\).

(ii) \((\mathcal{R}(A) \times 1 \sim) \cup (1 \sim \times \mathcal{R}(B)) = 1 - (\mathcal{R}(A))^c \times (\mathcal{R}(B))^c\).

(iii) \(1 - \mathcal{R}(A) \times \mathcal{R}(B) = ((\mathcal{R}(A))^c \times 1 \sim) \cup (1 \sim \times (\mathcal{R}(B))^c)\).

Proof:

(i) Let \(\mathcal{R}(A) = (\mathcal{R}(A), \mathcal{R}(A))\), \(\mathcal{R}(B) = (\mathcal{R}(B), \mathcal{R}(B))\)

\[
\begin{align*}
\mathcal{R}(A) &= \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle \\
\mathcal{R}(A) &= \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle \\
\mathcal{R}(B) &= \langle u, \mu_{\mathcal{R}(B)}(u), \nu_{\mathcal{R}(B)}(u) \rangle \\
\mathcal{R}(B) &= \langle u, \mu_{\mathcal{R}(B)}(u), \nu_{\mathcal{R}(B)}(u) \rangle .
\end{align*}
\]

Since \(\mathcal{R}(A) \times 1 \sim = \langle u, \min(\mu_{\mathcal{R}(A)}, 1 \sim), \max(\nu_{\mathcal{R}(A)}(u), 0 \sim) \rangle = \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle = \mathcal{R}(A)\).

Likewise \(\mathcal{R}(A) \times 1 \sim = \mathcal{R}(A)\). Therefore we have

\[
\begin{align*}
(\mathcal{R}(A) \times 1 \sim) \cap (1 \sim \times \mathcal{R}(B)) &= ((\mathcal{R}(A) \times 1 \sim) \cap \mathcal{R}(B)) \times 1 \sim \\
(\mathcal{R}(A) \times 1 \sim) \cap (1 \sim \times \mathcal{R}(B)) &= (\mathcal{R}(A) \cap \mathcal{R}(B)) \times \mathcal{R}(A) \times \mathcal{R}(B) \\
(\mathcal{R}(A) \times 1 \sim) \cup (1 \sim \times \mathcal{R}(B)) &= 1 - (\mathcal{R}(A))^c \times (\mathcal{R}(B))^c.
\end{align*}
\]

(ii) \((\mathcal{R}(A) \times 1 \sim) \cup (1 \sim \times \mathcal{R}(B)) = ((\mathcal{R}(A) \times 1 \sim) \cup (1 \sim \times \mathcal{R}(B))) \\
\mathcal{R}(A) \times 1 \sim \cup (1 \sim \times \mathcal{R}(B)) = ((\mathcal{R}(A) \times 1 \sim) \cup (1 \sim \times \mathcal{R}(B))) \\
\mathcal{R}(A) \times 1 \sim \cup (1 \sim \times \mathcal{R}(B)) = ((\mathcal{R}(A) \times 1 \sim) \cup \mathcal{R}(A) \times \mathcal{R}(B)) = (\mathcal{R}(A))^c \times (\mathcal{R}(B))^c.
\]

(iii) Obvious by putting \(\mathcal{R}(A), \mathcal{R}(B)\) instead of \((\mathcal{R}(A))^c, (\mathcal{R}(B))^c\) in (ii)

Lemma 3.7: Let \(f : U_i \to V_i (i = 1, 2)\) be a functions and \(\mathcal{R}(A)\) and \(\mathcal{R}(B)\) intuitionistic fuzzy rough sets of \(V_1\) and \(V_2\) respectively then \((f_1 \times f_2)^{-1}(\mathcal{R}(A), \mathcal{R}(B)) = f_1^{-1}(\mathcal{R}(A)) \times f_2^{-1}(\mathcal{R}(B))\).

Proof: \(\mathcal{R}(A) = (\mathcal{R}(A), \mathcal{R}(A))\) and \(\mathcal{R}(B) = (\mathcal{R}(B), \mathcal{R}(B))\),

where

\[
\begin{align*}
\mathcal{R}(A) &= \langle u_1, \mu_{\mathcal{R}(A)}(u_1), \nu_{\mathcal{R}(A)}(u_1) \rangle \\
\mathcal{R}(A) &= \langle u_1, \mu_{\mathcal{R}(A)}(u_1), \nu_{\mathcal{R}(A)}(u_1) \rangle \\
\mathcal{R}(B) &= \langle u_2, \mu_{\mathcal{R}(B)}(u_2), \nu_{\mathcal{R}(B)}(u_2) \rangle \\
\mathcal{R}(B) &= \langle u_2, \mu_{\mathcal{R}(B)}(u_2), \nu_{\mathcal{R}(B)}(u_2) \rangle ,
\end{align*}
\]

for each \((u_1, u_2) \in U_1 \times U_2\) we have

\[
\begin{align*}
(f_1 \times f_2)^{-1}(\mathcal{R}(A), \mathcal{R}(B))(u_1, u_2) &= ((f_1 \times f_2)^{-1}(\mathcal{R}(A) \times \mathcal{R}(B))(u_1, u_2), \\
(f_1 \times f_2)^{-1}(\mathcal{R}(A) \times \mathcal{R}(B))(u_1, u_2) &= (\mathcal{R}(A) \times \mathcal{R}(B))(f_1 \times f_2)(u_1, u_2).
\end{align*}
\]
Lemma 3.10: Let f be a map from a set $U$ to a set $V$ and let $\mathcal{R}(A) = (\mathcal{R}(A), \overline{\mathcal{R}}(A))$ be an intuitionistic fuzzy rough set of $U$. Then the $\mathcal{R}(A)$ graph of f is defined to be an intuitionistic fuzzy rough set $g(\mathcal{R}(A))=\langle u \times v, g(\mathcal{R}(A)), g(\overline{\mathcal{R}}(A)) \rangle$ of $U \times V$ where $g(\mathcal{R}(A)) = (g(\mu_{\mathcal{R}(A)}), g(\nu_{\mathcal{R}(A)}))$ and $g(\overline{\mathcal{R}}(A)) = (g(\mu_{\overline{\mathcal{R}}(A)}), g(\nu_{\overline{\mathcal{R}}(A)}))$

\textbf{Proof:} Let $\mathcal{R}(A) = (\mathcal{R}(A), \overline{\mathcal{R}}(A))$ and $\mathcal{R}(B) = (\mathcal{R}(B), \overline{\mathcal{R}}(B))$, where

$\mathcal{R}(A) = \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle$
$\overline{\mathcal{R}}(A) = \langle u, \mu_{\overline{\mathcal{R}}(A)}(u), \nu_{\overline{\mathcal{R}}(A)}(u) \rangle$
and
$\mathcal{R}(B) = \langle v, \mu_{\mathcal{R}(B)}(v), \nu_{\mathcal{R}(B)}(v) \rangle$
$\overline{\mathcal{R}}(B) = \langle v, \mu_{\overline{\mathcal{R}}(B)}(v), \nu_{\overline{\mathcal{R}}(B)}(v) \rangle$

For each $u \in U$ we have $g^{-1}(\mathcal{R}(A) \times \mathcal{R}(B))(u) = (\mathcal{R}(A) \times \mathcal{R}(B))g(u) = (\mathcal{R}(A) \times \mathcal{R}(B))(u, f(u)) = \langle u, f(u), min(\mu_{\mathcal{R}(B)}(u), \mu_{\mathcal{R}(B)}(f(u)), max(\nu_{\mathcal{R}(B)}(u), \nu_{\mathcal{R}(B)}(f(u))) \rangle$

Similarly $g^{-1}(\mathcal{R}(A) \times \mathcal{R}(B)) = (\mathcal{R}(A) \cap f^{-1}(\mathcal{R}(B)))(u)$. Therefore $g^{-1}(\mathcal{R}(A) \times \mathcal{R}(B))(u) = (\mathcal{R}(A) \cap f^{-1}(\mathcal{R}(B)))(u)$.

Lemma 3.10: Let $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(C), \mathcal{R}(D)$ be intuitionistic fuzzy rough set in $U$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B), \mathcal{R}(C) \subseteq \mathcal{R}(D)$ then $\mathcal{R}(A) \times \mathcal{R}(C) \subseteq \mathcal{R}(B) \times \mathcal{R}(D)$.

\textbf{Proof:} Let $\mathcal{R}(A) = (\mathcal{R}(A), \overline{\mathcal{R}}(A))$, $\mathcal{R}(B) = (\mathcal{R}(B), \overline{\mathcal{R}}(B))$, $\mathcal{R}(C) = (\mathcal{R}(C), \overline{\mathcal{R}}(C))$, $\mathcal{R}(D) = (\mathcal{R}(D), \overline{\mathcal{R}}(D))$ be intuitionistic fuzzy rough sets with $\mathcal{R}(A) = \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle$
$\overline{\mathcal{R}}(A) = \langle u, \mu_{\overline{\mathcal{R}}(A)}(u), \nu_{\overline{\mathcal{R}}(A)}(u) \rangle$
$\mathcal{R}(B) = \langle v, \mu_{\mathcal{R}(B)}(v), \nu_{\mathcal{R}(B)}(v) \rangle$
$\overline{\mathcal{R}}(B) = \langle v, \mu_{\overline{\mathcal{R}}(B)}(v), \nu_{\overline{\mathcal{R}}(B)}(v) \rangle$

and $\mathcal{R}(A) = \langle u, \mu_{\mathcal{R}(A)}(u), \nu_{\mathcal{R}(A)}(u) \rangle$
$\overline{\mathcal{R}}(A) = \langle u, \mu_{\overline{\mathcal{R}}(A)}(u), \nu_{\overline{\mathcal{R}}(A)}(u) \rangle$
$\mathcal{R}(B) = \langle v, \mu_{\mathcal{R}(B)}(v), \nu_{\mathcal{R}(B)}(v) \rangle$
$\overline{\mathcal{R}}(B) = \langle v, \mu_{\overline{\mathcal{R}}(B)}(v), \nu_{\overline{\mathcal{R}}(B)}(v) \rangle$

Since $\mathcal{R}(A) \subseteq \mathcal{R}(B) \Rightarrow (\mathcal{R}(A), \overline{\mathcal{R}}(A)) \subseteq (\mathcal{R}(B), \overline{\mathcal{R}}(B)) \Rightarrow \mu_{\mathcal{R}(A)} \subseteq \mu_{\mathcal{R}(B)}, \nu_{\mathcal{R}(A)} \subseteq \nu_{\mathcal{R}(B)}$ also
\[ \mathcal{R}(C) \subseteq \mathcal{R}(D) \Rightarrow (\overline{\mathcal{R}(C)}, \overline{\mathcal{R}(D)}) \subseteq (\overline{\mathcal{R}(D)}, \overline{\mathcal{R}(D)}) \Rightarrow \mu_{\overline{\mathcal{R}(C)}} \subseteq \mu_{\overline{\mathcal{R}(D)}} \]
\[ \subseteq \mu_{\overline{\mathcal{R}(D)}}, \nu_{\overline{\mathcal{R}(C)}} \subseteq \nu_{\overline{\mathcal{R}(D)}}, \text{we have } \min(\mu_{\overline{\mathcal{R}(A)}}, \nu_{\overline{\mathcal{R}(C)}}) \leq \min(\mu_{\overline{\mathcal{R}(B)}}, \mu_{\overline{\mathcal{R}(D)}}) \text{and} \]
\[ \max(\nu_{\overline{\mathcal{R}(B)}}, \nu_{\overline{\mathcal{R}(D)}}) \geq \max(\nu_{\overline{\mathcal{R}(B)}}, \nu_{\overline{\mathcal{R}(D)}}) \]

4. Intuitionistic fuzzy rough topology

**Definition 4.1:** Let \( U \) be an universe, \( R \) be an intuitionistic fuzzy equivalence relation on \( U \) and \( \tau_R(A) = \{1 \sim, 0 \sim, \overline{\mathcal{R}(A)}, \mathcal{R}(A)\} \) where \( A \subseteq U \). Then \( \tau_R(A) \) satisfies the following axioms

1. \( 0 \sim \) and \( 1 \sim \in \tau_R(A) \).
2. The union of the elements of any subcollection of \( \tau_R(A) \) is in \( \tau_R(A) \).
3. The intersection of the elements of any finite subcollection of \( \tau_R(A) \) is in \( \tau_R(A) \).

That is \( \tau_R(A) \) forms a topology on \( U \) called the intuitionistic fuzzy rough topology with respect to \( A \). We call \( (U, \tau_R(A)) \) as intuitionistic fuzzy rough topological space. The elements of \( \tau_R(A) \) are called intuitionistic fuzzy rough open sets.

In this case the pair \( (U, \tau_R(A)) \) is called as intuitionistic fuzzy rough topological space (IFRTS for short) and any IFRS in \( \tau_R(A) \) is known as (IFROS) in \( U \).

**Remark 4.2:** Let \( U \) be an universe of objects and \( R \) be an equivalence relation on \( U \). For \( X \subseteq U \), let \( \tau_R(X) = \{0 \sim, 1 \sim, \overline{\mathcal{R}(X)}, \mathcal{R}(X)\} \). We note that \( 1 \sim \) and \( 0 \sim \in \tau_R(X) \). Since \( \mathcal{R}(X) \cup \overline{\mathcal{R}(X)} \subseteq \overline{\mathcal{R}(X)} \subseteq \tau_R(X) \) and \( \overline{\mathcal{R}(X)} \cap \mathcal{R}(X) \subseteq \overline{\mathcal{R}(X)} \in \tau_R(X) \).

**Remark 4.3:** The rough topology thus obtained is the smallest intuitionistic fuzzy topology since it contains only \( 0 \sim, 1 \sim \), \( \overline{\mathcal{R}(A)} \) and \( \mathcal{R}(A) \).

**Example 4.4:** Let \( (U, R) \) be an intuitionistic fuzzy approximation space where \( U = \{x_1, x_2, x_3\} \) and \( R \in R(U \times U) \) is defined as follows
\[ R = \{(x_1, x_1), 1 \sim, 0 \sim, (x_1, x_2), 0.4, 0.5\} \]
\[ (x_2, x_2), 1 \sim, 0 \sim, (x_2, x_3), 0.4, 0.3\} \]
\[ (x_3, x_3), 1 \sim, 0 \sim, (x_1, x_3), 0.5, 0.4\} \]
\[ (x_3, x_3), 1 \sim, 0 \sim, (x_1, x_3), 0.5, 0.4\} \]

Let \( A = \{(x_1, 0.6, 0.2), (x_2, 0.5, 0.4), (x_3, 0.5, 0.4)\} \) be an intuitionistic fuzzy set on \( U \) then by definition we have
\[ \mu_{\overline{\mathcal{R}(A)}}(x_1) = 0.6, \mu_{\overline{\mathcal{R}(A)}}(x_2) = 0.5, \mu_{\overline{\mathcal{R}(A)}}(x_3) = 0.5. \]
\[ \nu_{\overline{\mathcal{R}(A)}}(x_1) = 0.2, \nu_{\overline{\mathcal{R}(A)}}(x_2) = 0.4, \nu_{\overline{\mathcal{R}(A)}}(x_3) = 0.4. \]
\[ \mu_{\mathcal{R}(A)}(x_1) = 0.5, \mu_{\mathcal{R}(A)}(x_2) = 0.5, \mu_{\mathcal{R}(A)}(x_3) = 0.5. \]
\[ \nu_{\mathcal{R}(A)}(x_1) = 0.4, \nu_{\mathcal{R}(A)}(x_2) = 0.4, \nu_{\mathcal{R}(A)}(x_3) = 0.4. \]
\[ \mathcal{R}(A) = \{(x_1, 0.6, 0.2), (x_2, 0.5, 0.4), (x_3, 0.5, 0.4)\} \].
\[ R(A) = \{ (x_1, 0.5, 0.4), (x_2, 0.5, 0.4), (x_3, 0.5, 0.4) \}. \]
\[ \Re(A) = \{ (x_1(0.5, 0.4), (0.6, 0.2)), (x_2, (0.5, 0.4), (0.5, 0.4)), (x_3, (0.5, 0.4), (0.5, 0.4)) \}. \]

Example 4.5: Let \((U, R)\) be an intuitionistic fuzzy approximation space where \(U = \{ x_1, x_2, x_3 \}\) and \(R \in R(U \times U)\) is defined as follows
\[ R = \{ ((x_1, x_1), 1 \sim, 0 \sim), ((x_1, x_2), 0.4, 0.5), ((x_2, x_1), 0.4, 0.5), \]
\[ ((x_2, x_2), 1 \sim, 0 \sim), ((x_2, x_3), 0.4, 0.3), ((x_3, x_2), 0.4, 0.3), \]
\[ ((x_3, x_3), 1 \sim, 0 \sim), ((x_1, x_3), 0.5, 0.4), ((x_3, x_1), 0.5, 0.4) \}. \]

Let \(A = \{ (x_1, 0.5, 0.3), (x_2, 0.5, 0.5), (x_3, 0.5, 0.4) \}\) be an intuitionistic fuzzy set on \(U\) then by
definition we have
\[ \mu_{\Re(A)}(x_1) = 0.5, \mu_{\Re(A)}(x_2) = 0.5, \mu_{\Re(A)}(x_3) = 0.5. \]
\[ \nu_{\Re(A)}(x_1) = 0.3, \nu_{\Re(A)}(x_2) = 0.4, \nu_{\Re(A)}(x_3) = 0.4. \]
\[ \mu_{R(A)}(x_1) = 0.5, \mu_{R(A)}(x_2) = 0.5, \mu_{R(A)}(x_3) = 0.5. \]
\[ \nu_{R(A)}(x_1) = 0.4, \nu_{R(A)}(x_2) = 0.5, \nu_{R(A)}(x_3) = 0.4. \]
\[ \Re(A) = \{ (x_1, 0.5, 0.3), (x_2, 0.5, 0.4), (x_3, 0.5, 0.4) \}. \]
\[ R(A) = \{ (x_1, 0.5, 0.4), (x_2, 0.5, 0.4), (x_3, 0.5, 0.4) \}. \]
\[ \Re(A) = \{ (x_1(0.5, 0.4), (0.5, 0.3)), (x_2, (0.5, 0.5), (0.5, 0.4)), (x_3, (0.5, 0.4), (0.5, 0.4)) \}. \]

Proposition 4.6:
1. If \(R(A) = 0 \sim\) and \(\Re(A) = 1 \sim\) then \(\tau_{\Re}(A) = \{ 0 \sim, 1 \sim \}\) the indiscrete intuitionistic fuzzy rough topology on \(U\).
2. If \(R(A) = 0 \sim\) and \(\Re(A) = A\) then the intuitionistic fuzzy rough topology \(\tau_{\Re}(A) = \{ 0 \sim, 1 \sim, R(A) \}\).
3. If \(R(A) = 0 \sim\) and \(\Re(A) \neq 1 \sim\) then the intuitionistic fuzzy rough topology \(\tau_{\Re}(A) = \{ 0 \sim, 1 \sim, \Re(A) \}\).
4. If \(R(A) \neq 0 \sim\) and \(\Re(A) = 1 \sim\) then the intuitionistic fuzzy rough topology \(\tau_{\Re}(A) = \{ 0 \sim, 1 \sim, R(A) \}\).
5. If \(R(A) \neq \Re(A)\) where \(R(A) \neq 0 \sim\) and \(\Re(A) \neq 1 \sim\) then \(\tau_{\Re}(A) = \{ 0 \sim, 1 \sim, R(A), \Re(A) \}\) is the discrete intuitionistic fuzzy rough topology on \(U\).

Theorem 4.7: Let \(U\) be a non-empty finite universe and \(A \subseteq U\). Let \(\tau_{\Re}(A)\) be the \(\ReT\) on \(U\) with respect to \(A\). Then \(\tau_{\Re}(A)^c\) is a topology on \(U\).

Proof: The intuitionistic fuzzy rough topology on \(U\) with respect to \(A\) is given by
\[ \tau_{\Re}(A) = \{ 0 \sim, 1 \sim, R(A), \Re(A) \} \]
\[ = \{ 0 \sim, 1 \sim, \langle x, \mu_{\Re(A)}, \nu_{\Re(A)} \rangle, \langle x, \mu_{\Re(A)} \cdot \nu_{\Re(A)} \rangle \}. \]

Therefore \(\tau_{\Re}(A)^c = \{ 0 \sim, 1 \sim, [R(A)]^c, [\Re(A)]^c \} \)
\[ = \{ 1 \sim, 0 \sim, \langle x, \nu_{\Re(A)}, \mu_{\Re(A)} \rangle, \langle x, \nu_{\Re(A)} \cdot \mu_{\Re(A)} \rangle \}. \]
Consider \([\{R(A)\}^c \cup \{\overline{R}(A)\}^c]\)
\[= [R(A) \cap \overline{R}(A)]^c = [R(A)]^c \in [\tau_R(A)]^c.\] The arbitrary union of members of \([\tau_R(A)]^c\) are in \([\tau_R(A)]^c.\]

Also
\[([R(A)]^c \cap (\overline{R}(A)]^c] = [R(A) \cup \overline{R}(A)]^c = [\overline{R}(A)]^c \in [\tau_R(A)]^c.\] That is finite intersection of member of \([\tau_R(A)]^c\) are in \([\tau_R(A)]^c.\] Thus \([\tau_R(A)]^c\) is a IF\(\mathbb{R}\)T on \(U.\)

**Remark 4.8:** Elements of \([\tau_R(A)]^c\) are called IF\(\mathbb{R}\) closed sets.

**Definition 4.9:** If \((U, \tau_R(X))\) is an intuitionistic fuzzy rough topological space with respect to \(X\) where \(X \subseteq U\) and if \(A \subseteq U\), then the intuitionistic fuzzy rough interior of \(A\) is defined as the union of all intuitionistic fuzzy rough open subsets contained in \(A\) and is denoted by \(\mathbb{R}\text{int}(A)\).

\((i.e)\)
\[
\mathbb{R}\text{int}(A) = \cup \{G: G \text{ is an IF\(\mathbb{R}\)OS in } U \text{ and } G \subseteq A\}.
\]
That is \(\mathbb{R}\text{int}(A)\) is the largest intuitionistic fuzzy rough open subset of \(A\).

The intuitionistic fuzzy rough closure of \(A\) is defined as the intersection of all intuitionistic fuzzy rough closed subsets containing \(A\) and is denoted by \(\mathbb{R}\text{cl}(A)\).

\((i.e)\)
\[
\mathbb{R}\text{cl}(A) = \cap \{K: K \text{ is an IF\(\mathbb{R}\)CS in } U \text{ and } A \subseteq K\}.
\]
That is \(\mathbb{R}\text{cl}(A)\) is the smallest intuitionistic fuzzy rough closed set containing \(A\).

**Proposition 4.10:** Let \(\mathbb{R}(A)\) and \(\mathbb{R}(B)\) be two IF\(\mathbb{R}\)T spaces in \(X\) then

1. \(\mathbb{R}(A) = \mathbb{R}(B)\) if and only if \(R(A) = \overline{R}(B)\) and \(\overline{R}(A) = \overline{R}(B)\).

2. \(\mathbb{R}(A) \subseteq \mathbb{R}(B)\) if and only if \(R(A) \subseteq \overline{R}(B)\) and \(IF\(\mathbb{R}\)(A) \subseteq \overline{R}(B)\).

\((i.e)\)
\[
\mu_{\overline{R}(A)} \subseteq \mu_{\overline{R}(B)} \text{ and } \nu_{\overline{R}(A)} \subseteq \nu_{\overline{R}(B)}
\]
\[
\mu_{\overline{R}(A)} \subseteq \mu_{\overline{R}(B)} \text{ and } \nu_{\overline{R}(A)} \subseteq \nu_{\overline{R}(B)}
\]

3. \([\mathbb{R}(A)]^{c^c} = \mathbb{R}(A)\).

4. \(\mathbb{R}(A) \cup \mathbb{R}(B) \subseteq \mathbb{R}(A \cup B)\).

5. \(\mathbb{R}(A) \cap \mathbb{R}(B) \supseteq \mathbb{R}(A \cap B)\).

6. \(\mathbb{R}(A) \supseteq \mathbb{R}(B)\) if and only if \(\mu_{\overline{R}(A)} \cup \nu_{\overline{R}(A)} \supseteq \mu_{\overline{R}(B)} \cup \nu_{\overline{R}(B)}\) and \(\mu_{\overline{R}(A)} \cup \nu_{\overline{R}(A)} \supseteq \mu_{\overline{R}(B)} \cup \nu_{\overline{R}(B)}\).

7. \([\mathbb{R}(A)]^{c^c} = \{x, (\nu_{\overline{R}(A)}), (\mu_{\overline{R}(A)}), (\nu_{\overline{R}(A)}), (\mu_{\overline{R}(A)})\}\)

8. \(\cup \mathbb{R}(A) = \{x, (\cup \mu_{\overline{R}(A)}), (\cup \nu_{\overline{R}(A)}), (\cup \mu_{\overline{R}(A)}), (\cup \nu_{\overline{R}(A)})\}\).

9. \(\cap \mathbb{R}(A) = \{x, (\cap \mu_{\overline{R}(A)}), (\cap \nu_{\overline{R}(A)}), (\cap \mu_{\overline{R}(A)}), (\cap \nu_{\overline{R}(A)})\}\).

10. \(\mathbb{R}(A) - \mathbb{R}(B) = \mathbb{R}(A \cap (\mathbb{R}(B))^{c^c})\).
11. $\mathbb{R}(A) \subseteq \mathbb{R}(B) \iff (\mathbb{R}(A))^c \subseteq (\mathbb{R}(B))^c$.

**Proposition 4.11**: Let $(U, \tau_R(X))$ be an intuitionistic fuzzy rough topological space with respect to $X$ where $X \subseteq U$. Let $A, B \subseteq U$. Then

1. $A \subseteq \text{Rel}(A)$.
2. $A$ is intuitionistic fuzzy rough closed if and only if $\text{Rel}(A) = A$.
3. $\text{Rel}(0^\sim) = 0^\sim$ and $\text{Rel}(1^\sim) = 1^\sim$.
4. $A \subseteq B \Rightarrow \text{Rel}(A) \subseteq \text{Rel}(B)$.
5. $\text{Rel}(A \cup B) = \text{Rel}(A) \cup \text{Rel}(B)$.
6. $\text{Rel}(A \cap B) \subseteq \text{Rel}(A) \cap \text{Rel}(B)$.
7. $\text{Rel}(\text{Rel}(A)) = \text{Rel}(A)$.

**Proof:**

1. By definition of intuitionistic fuzzy rough closure, $A \subseteq \text{Rel}(A)$.
2. If $A$ is intuitionistic fuzzy rough closed, then $A$ is the smallest intuitionistic fuzzy rough closed set containing itself and hence $\text{Rel}(A) = A$. Conversely, if $\text{Rel}(A) = A$, then $A$ is the smallest intuitionistic fuzzy rough closed set containing itself and hence $A$ is intuitionistic fuzzy rough closed.
3. Since $0^\sim$ and $1^\sim$ are intuitionistic fuzzy rough closed in $(U, \tau_R(X))$, $\text{Rel}(0^\sim) = 0^\sim$ and $\text{Rel}(1^\sim) = 1^\sim$.
4. When $A \subseteq B$, since $B \subseteq \text{Rel}(B)$, $A \subseteq \text{Rel}(B)$. That is $\text{Rel}(B)$ is a intuitionistic fuzzy rough closed set containing $A$. But $\text{Rel}(A)$ is the smallest intuitionistic fuzzy rough closed set containing $A$. Therefore, $\text{Rel}(A) \subseteq \text{Rel}(B)$.
5. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $\text{Rel}(A) \subseteq \text{Rel}(A \cup B)$ and $\text{Rel}(B) \subseteq \text{Rel}(A \cup B)$. Therefore, $\text{Rel}(A) \cup \text{Rel}(B) \subseteq \text{Rel}(A \cup B)$. Since $A \cup B \subseteq \text{Rel}(A) \cup \text{Rel}(B)$, and since $\text{Rel}(A \cup B)$ is the smallest intuitionistic fuzzy rough closed set containing $A \cup B$, $\text{Rel}(A \cup B) \subseteq \text{Rel}(A) \cup \text{Rel}(B)$. Thus $\text{Rel}(A \cup B) = \text{Rel}(A) \cup \text{Rel}(B)$.
6. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\text{Rel}(A \cap B) \subseteq \text{Rel}(A) \cap \text{Rel}(B)$.
7. Since $\text{Rel}(A)$ is intuitionistic fuzzy rough closed, $\text{Rel}(\text{Rel}(A)) = \text{Rel}(A)$.

**Theorem 4.12**: (Kuratowski closure axioms)
The intuitionistic fuzzy rough closure in a intuitionistic fuzzy rough topological space $(U, \tau_R(X))$ where $X \subseteq U$ is a Kuratowski closure operator.

**Proof**: By proposition [5.10] intuitionistic fuzzy rough closure is a mapping of $\text{P}(U)$ into itself satisfying the following Kuratowski closure axioms that is

1. $\text{Rel}(0^\sim) = 0^\sim$.
2. \( A \subseteq \mathbb{R} \text{cl}(A) \).
3. \( \mathbb{R} \text{cl}(A \cup B) = \mathbb{R} \text{cl}(A) \cup \mathbb{R} \text{cl}(B) \).
4. \( \mathbb{R} \text{cl}(\mathbb{R} \text{cl}(A)) = \mathbb{R} \text{cl}(A) \) where \( A, B \subseteq U \) and hence intuitionistic fuzzy rough closure is a Kuratowski closure axiom.

**Proposition 4.13:** If \((U, \tau R(X))\) is an intuitionistic fuzzy rough topological space with respect to \( X \) where \( X \subseteq U \) and \( A, B \) are subsets of \( U \) then

1. \( A \) is intuitionistic fuzzy rough open if and only if \( \mathbb{R} \text{int}(A) = A \).
2. \( \mathbb{R} \text{int}(0 \sim) = 0 \sim \), \( \mathbb{R} \text{int}(1 \sim) = 1 \sim \).
3. \( A \subseteq B \implies \mathbb{R} \text{int}(A) \subseteq \mathbb{R} \text{int}(B) \).
4. \( \mathbb{R} \text{int}(A) \cup \mathbb{R} \text{int}(B) \subseteq \mathbb{R} \text{int}(A \cup B) \).
5. \( \mathbb{R} \text{int}(A \cap B) = \mathbb{R} \text{int}(A) \cap \mathbb{R} \text{int}(B) \).
6. \( \mathbb{R} \text{int}(\mathbb{R} \text{int}(A)) = \mathbb{R} \text{int}(A) \).

**Definition 4.14:** Let \((U, \tau_U)\) and \((V, \sigma_V)\) be intuitionistic fuzzy rough topological spaces. The intuitionistic fuzzy rough product space (IF\( \mathbb{R} \)PTS, for short) of \((U, \tau_U)\) and \((V, \sigma_V)\) is the cartesian product of \( U \times V \) of intuitionistic fuzzy sets \( U \) and \( V \) together with the intuitionistic fuzzy rough topology \( \tau_U \times \tau_V \), generated by the family \( \{ p^{-1}_1(\mathbb{R}(A_i)), p^{-1}_2(\mathbb{R}(B_j)) : \mathbb{R}(A_i) \in \tau_U, \mathbb{R}(B_j) \in \sigma_V \} \) and \( p_1, p_2 \) are projections of \( U \times V \) onto \( U \) and \( V \) respectively. (i.e.) the family \( \{ p^{-1}_1(\mathbb{R}(A_i)), p^{-1}_2(\mathbb{R}(B_j)) : \mathbb{R}(A_i) \in \tau_U, \mathbb{R}(B_j) \in \sigma_V \} \) forms a sub base for intuitionistic fuzzy rough topology \( \tau_U \times \tau_V \) on \( U \times V \).

**Remark 4.15:** In the above definition, since \( p^{-1}_1(\mathbb{R}(A_i)) = \mathbb{R}(A_i) \times 1 \sim \) and \( p^{-1}_2(\mathbb{R}(B_j)) = 1 \sim \times \mathbb{R}(B_j) \) and \( (\mathbb{R}(A_i) \times 1 \sim) \cap (1 \sim \times \mathbb{R}(B_j)) = \mathbb{R}(A_i) \times \mathbb{R}(B_j) \), the family \( \beta = \{ \mathbb{R}(A_i) \times \mathbb{R}(B_j) \} : \mathbb{R}(A_i) \in \tau_U, \mathbb{R}(B_j) \in \sigma_V \} \) forms a base for the intuitionistic fuzzy rough product topological space \( \tau_U \times \tau_V \) of \( U \times V \).

**Lemma 4.16:** Let \( p_u \) and \( p_v \) be projection maps of \( U \times V \) onto \( U \) and \( V \) respectively. Suppose that \( A \) is an intuitionistic fuzzy rough topological space of \( U \) and \( B \) is an intuitionistic fuzzy rough topological space of \( V \). Then

(a) \( p^{-1}_u(A) = \mathbb{R}(A) \times 1 \sim \).
(b) \( p^{-1}_v = 1 \sim \times \mathbb{R}(B) \).

**Lemma 4.17:** Let \( \mathbb{R}(A) \) be an intuitionistic fuzzy rough closed set of an intuitionistic fuzzy topological space \( U \) and \( B \) be an intuitionistic fuzzy closed rough set of an intuitionistic fuzzy rough topological space \( V \). Then \( \mathbb{R}(A) \times \mathbb{R}(B) \) is an intuitionistic fuzzy rough closed set of the intuitionistic fuzzy rough product space \( U \times V \).

**Theorem 4.18:** If \( \mathbb{R}(A) \) and \( \mathbb{R}(B) \) be intuitionistic fuzzy rough sets of intuitionistic fuzzy rough topological space \( U \) and \( V \) respectively then
(i) \( cl(\mathbb{R}(A)) \times cl(\mathbb{R}(B)) \supseteq cl(\mathbb{R}(A)) \times cl(\mathbb{R}(B)). \)

(ii) \( int(\mathbb{R}(A)) \times int(\mathbb{R}(B)) \subseteq int(\mathbb{R}(A) \times \mathbb{R}(B)). \)

**Proof**

(i) Since \( \mathbb{R}(A) \subseteq \text{Rcl}(A) \) and \( \mathbb{R}(B) \subseteq \text{Rcl}(B) \) hence \( \mathbb{R}(A) \times \mathbb{R}(B) \subseteq \text{Rcl}(A) \times \text{Rcl}(B) \Rightarrow \text{Rcl}(\mathbb{R}(A) \times \mathbb{R}(B)) \subseteq \text{Rcl}(\mathbb{R}(A) \times \mathbb{R}(B)) \) from lemma 4.7 we have \( \text{Rcl}(\mathbb{R}(A) \times \mathbb{R}(B)) \subseteq \text{Rcl}(\mathbb{R}(A) \times \mathbb{R}(B)). \)

(ii) (ii) follows from (i) and the fact \( 1 \sim -\text{Rcl}(A) = int(1 - \mathbb{R}(A)) \)

**References**


