Empirical likelihood confidence intervals for the differences of quantiles

with missing data

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ABSTRACT: Suppose there are two nonparametric populations \( X \) and \( Y \) with missing data on both of them. Random imputation is used to fill in missing data, so the “complete” samples of \( X \) and \( Y \) can be constructed. Then the empirical likelihood confidence intervals for the differences of quantile are constructed.

Key words: Empirical likelihood; Confidence interval; Quantile; Missing data; Imputation.

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INTRODUCTION

Use \( F \) and \( G \) to denote the distribution functions of two nonparametric populations \( X \) and \( Y \) respectively. For any distribution function \( H(\cdot) \), define its \( q \)-th quantile by \( H^{-1}(q) = \inf \{ x \mid H(x) \geq q \} \) where \( 0 < q < 1 \). In this paper, we are interested in constructing confidence interval for the quantile differences

\[ \Delta = G^{-1}(q) - F^{-1}(q) \]

for fixed \( 0 < q < 1 \) with missing data on both of \( X \) and \( Y \). We assume that the \( q \)-th quantile of \( F \) and \( G \) are unique. Let \( \theta \) be the \( q \)-th quantile of \( F \). Then \( \theta + \Delta \) is the \( q \)-th quantile of \( G \). In the case of complete observations, related work can be found in [1]. But in fact, missing responses are common because all kinds of reasons, such as loss of information caused by uncontrollable factors, failure on the part of the investigator to gather correct information and so forth. In this situation, the usual inference procedures cannot be applied directly, so a common method to handle samples with missing data is to impute a value for each missing value and then obtain a ‘complete sample’. Then we use usual statistical methods to make inference based on the ‘complete sample’. Commonly used imputation methods include deterministic imputation and random imputation in [2-4].

On the other hand, Owen made a systematic study of the empirical likelihood method in the complete data settings in [5]. It has several advantages over the normal-approximation based methods and the bootstrap in constructing confidence intervals. EL intervals are range preserving and transformation respecting. Also, the shape and orientation of EL intervals are determined entirely by the data. And in [6-7], Wang and Rao first use EL method to model with missing data (they do not specify the form of the error distribution in the linear model, i.e the error distribution is nonparametric). Wang and Rao then use EL method to construct confidence intervals for the mean of the response variable in a nonparametric regression model with missing data. In [8]
Qin and Zhang constructed the semi-empirical likelihood confidence intervals on the quantile differences of \(X\) and \(Y\) with random hot deck imputation, and conducted a small simulation study to show that random hot deck imputation can improve the cover accuracy of confidence intervals. In this paper, we are interested in constructing empirical likelihood confidence intervals on \(\Delta\) in the case of incomplete observations with data described below.

Consider the following simple random samples of incomplete data associated with populations \((x_i, \delta_x), i = 1, 2, \cdots, m;\) and \((y_j, \delta_y), j = 1, 2, \cdots, n\), where

\[
\delta_{x,i} = \begin{cases} 
0, & \text{if } x_i \text{ is missing} \\
1, & \text{if } x_i \text{ is not missing} 
\end{cases},
\delta_{y,j} = \begin{cases} 
0, & \text{if } y_j \text{ is missing} \\
1, & \text{if } y_j \text{ is not missing} 
\end{cases}
\]

Throughout this paper, we assume that \(X\) and \(Y\) are missing completely at random (MCAR), i.e.

\[
p(\delta_{x,i} = 1 \mid x) = P(\delta_{x,i} = 1) = p_1 (\text{const} \tan t), \quad p(\delta_{y,j} = 1 \mid y) = P(\delta_{y,j} = 1) = p_2 (\text{const} \tan t) .
\]

We also assume that \((x_i, \delta_x), i = 1, 2, \cdots, m;\) and \((y_j, \delta_y), j = 1, 2, \cdots, n\) are independent.

Let

\[
r_x = \sum_{i=1}^{m} \delta_{x,i}, \quad r_y = \sum_{j=1}^{n} \delta_{y,j}, \quad m_x = m - r_x, \quad m_y = m - r_y,
\]

\[
S_{xx} = \{i: \delta_{x,i} = 1 \mid i = 1, 2, \cdots, m\}, \quad S_{mx} = \{i: \delta_{x,i} = 0 \mid i = 1, 2, \cdots, m\},
\]
\[
S_{yy} = \{j: \delta_{y,j} = 1 \mid j = 1, 2, \cdots, n\}, \quad S_{my} = \{j: \delta_{y,j} = 0 \mid j = 1, 2, \cdots, n\},
\]

Similar to [8],[10] we use fractional random hot deck imputation method to impute the missing values. We do not use the deterministic imputation as it does not proper in making inference for distribution functions(see [11]).

Let \(x_{il}\) and \(y_{jl}\) \((l = 1, 2, \cdots, k)\) be the imputed values for the missing data respectively, where \(x_{il}\) and \(y_{jl}\) \((l = 1, 2, \cdots, k)\) come from the responses of populations \(X\) and \(Y\) respectively at random. Let

\[
x_{ij} = \delta_{x,i} x_i + \frac{1 - \delta_{x,i}}{k} \sum_{l=1}^{k} x_{il}, \quad y_{ij} = \delta_{y,j} y_j + \frac{1 - \delta_{y,j}}{k} \sum_{l=1}^{k} y_{jl}, \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, n
\]

which represent ‘complete’ data after fractional random hot deck imputation.

In this paper, we investigate the asymptotic properties of the empirical likelihood ratio statistic for \(\Delta\) based on \(x_{ij}, y_{ij}\). The results are used to constructed asymptotic confidence intervals for \(\Delta\).

The rest of this paper is organized as follows. In section 2, the empirical likelihood ratio statistic is constructed, the limiting distribution of the statistic is given, and the empirical likelihood based confidence intervals for \(\Delta\) is constructed. The proof of main results are given in the section 3.

**MAIN RESULTS**
In this paper, we take bandwidths $a = a_m > 0$, $b = b_n > 0$, and kernels $K_1$ and $K_2$, where $a \to 0$ as $m \to \infty$ and $b \to 0$ as $n \to \infty$. Define

$$G_1(t) = \int_{-\infty}^{a} K_1(u) du, \quad G_2(t) = \int_{-\infty}^{b} K_2(u) du,$$

$$\omega_{li}(x_i ; x_{ij}, \theta, \Delta) = \delta_i G_1(\theta - x_i) + \frac{1 - \delta_i}{k} \sum_{j=1}^{k} G_1(\theta - x_{ij}) - q, \quad i = 1, 2, \ldots, m$$

$$\omega_{jn}(y_j ; y_{jl}, \theta, \Delta) = \delta_j G_2(\theta - y_j) + \frac{1 - \delta_j}{k} \sum_{l=1}^{k} G_2(\theta + \Delta - y_{jl}) - q, \quad j = 1, 2, \ldots, n$$

Similar to [1] and [9], the empirical likelihood function is defined as

$$\prod_{i=1}^{m} p_i \prod_{j=1}^{n} q_j$$

(2.1)

Where $p_i > 0, i = 1, 2, \ldots, m, \sum_{i} p_i = 1$, and $q_j > 0, j = 1, 2, \ldots, n, \sum_{j} q_j = 1$. Define the log-empirical likelihood ratio statistic

$$R(\Delta) = \sup_{p_i > 0, q_j > 0, \sum_{i} p_i = 1, \sum_{j} q_j = 1} \left\{ \sum_{i=1}^{m} \log(mp_i) + \sum_{j=1}^{n} \log(nq_j) \right\} = \sup_{\theta} R(\Delta, \theta)$$

(2.2)

Where $R(\Delta, \theta) = \sup_{p_i > 0, j = 1, \ldots, m, q_j > 0, j = 1, \ldots, n} \left\{ \sum_{i=1}^{m} \log(mp_i) + \sum_{j=1}^{n} \log(nq_j) \right\}$ and $p_i, q_j$ are subject to restrictions:

$$p_i > 0, i = 1, 2, \ldots, m, \sum_{i} p_i = 1, \sum_{i} p_i \omega_{li}(x_i ; x_{ij}, \theta, \Delta) = 0$$

$$q_j > 0, j = 1, 2, \ldots, n, \sum_{j} q_j = 1, \sum_{j} q_j \omega_{jn}(y_j ; y_{jl}, \theta, \Delta) = 0$$

From Lagrange multipliers, we can show that

$$R(\Delta, \theta) = -\sum_{i=1}^{m} \log\{ 1 + \lambda_i(\theta) \omega_{li}(x_i ; x_{ij}, \theta, \Delta) \} - \sum_{j=1}^{n} \log\{ 1 + \lambda_j(\theta) \omega_{jn}(y_j ; y_{jl}, \theta, \Delta) \}$$

where $\lambda_j(\theta), j = 1, 2$ are determined by the following two equations:

$$\frac{1}{m} \sum_{i=1}^{m} \frac{\omega_{li}(x_i ; x_{ij}, \theta, \Delta)}{1 + \lambda_i(\theta) \omega_{li}(x_i ; x_{ij}, \theta, \Delta)} = 0, \quad \frac{1}{n} \sum_{j=1}^{n} \frac{\omega_{jn}(y_j ; y_{jl}, \theta, \Delta)}{1 + \lambda_j(\theta) \omega_{jn}(y_j ; y_{jl}, \theta, \Delta)} = 0$$

Let $\partial R(\Delta, \theta) / \partial \theta = 0$, we can obtain the empirical equation:

$$\frac{1}{m} \sum_{i=1}^{m} \frac{\alpha_{li}(x_i ; x_{ij}, \theta, \Delta)}{1 + \lambda_i(\theta) \omega_{li}(x_i ; x_{ij}, \theta, \Delta)} + \frac{1}{n} \sum_{j=1}^{n} \frac{\alpha_{ln}(y_j ; y_{jl}, \theta, \Delta)}{1 + \lambda_j(\theta) \omega_{ln}(y_j ; y_{jl}, \theta, \Delta)} = 0$$

(2.3)

where
Use $\theta_0$ to denote the true value of $\theta$. We make some assumptions in the following: **H 2.1** $\theta_0 \in \Omega$ and $\Omega$ is an open interval.

**H 2.2** Denote $f(t) = \partial F(t) / \partial t$ and $g(t) = \partial G(t) / \partial t$. For some $t_0 \geq 2$, suppose that

$f(t_0-1)(t)$ exists and is continuous in a neighborhood of $\theta_0$, and that $g(t_0-1)(t)$ exists and is continuous in a neighborhood of $\theta_0 + \Delta$. Assume that $f(\theta_0) g(\theta_0 + \Delta) > 0$.

**H 2.3** $\frac{n}{m} \to h(0 < h < \infty)$ as $m, n \to \infty$.

**H 2.4** For $i = 1, 2, K_i$ are bounded and satisfy Lipschitz condition of order 1: $K_i^{(2)}(\cdot)$ exists and are bounded. For $i = 1, 2$ assume that for some $c > 0$,

$$\int_{|u| > c} K_i(u) du = O(a^b), \quad \int K_i(u) du = O(b^b), \quad \int u^b K_i(u) du < \infty,$$

$$\int K_i(u) du < \infty, \quad \int u^j K_i(u) du = \begin{cases} 1, & j = 0 \\ 0, & 1 \leq j \leq t_0 - 1. \end{cases}$$

**H 2.5** There exists $r(1/3 < r < 1/2)$ such that

$$n'a^b \to 0, \quad n'b^b \to 0, \quad n'a \to \infty, \text{and} \quad n'b \to \infty, n'^2 a^b \to 0, \quad n'^2 b^b \to 0, \quad n, m \to \infty.$$

**Theorem 2.1** Suppose that assumptions (2.1) through (2.5) are satisfied. Then there exists a root $\theta_{m,n}$ of equation (2.3) such that $R(\Delta, \theta)$ attains its local maximum at $\theta_{m,n}$, and as $m,n \to \infty$,

$$\sqrt{m}(\theta_{m,n} - \theta_0) \xrightarrow{d} N(0, \frac{1}{c_0} \left\{ \left( 1 - p_2 + kp_2 \right)^2 f^2(\theta_0) \sigma_2^2 + \frac{h(1-p_1+ kp_1)^2 g^2(\theta_0 + \Delta)}{k^2} \sigma_0^2 \right\})$$

and

$$-2R(\Delta, \theta_{m,n}) \xrightarrow{d} \frac{\left( 1 - p_1 + p_1^{-1} \right) h g^2(\theta_0 + \Delta) + \left( 1 - p_2 + p_2^{-1} \right) f^2(\theta_0)}{f^2(\theta_0) + h g^2(\theta_0 + \Delta)}.$$

Where $\sigma_1^2 = \left( 1 - p_2 + p_2^{-1} \right) q(1-q)$, $\sigma_2^2 = \left( 1 - p_2 + p_2^{-1} \right) q(1-q)$.

It is interesting to notice that the empirical likelihood ratio under imputation is asymptotically distributed as a scaled chi-square variable. The reason for this deviation from the standard results is that the complete data after imputation are
dependent. Denote \( a_0(\Delta) = \frac{1}{c_0} \left\{ \frac{1}{k} \left( -\frac{P_1}{P_1} + \frac{P_1}{P_1} \right) h + \frac{1}{k} \left( -\frac{P_2}{P_2} + \frac{P_2}{P_2} \right) f^2(\Delta) \right\} \), so construct a confidence interval on \( \Delta \) by using above results, we need to get a consistent estimator of \( a_0(\Delta) \). \( P_1 \) and \( P_2 \) can be consistently by

\[
p_1 = \frac{1}{m} \sum_{i=1}^{m} \delta_{i_1} \quad \text{and} \quad n \sum_{j=1}^{n} \delta_{j_2} \quad \text{respectively},
\]

\( \hat{h} \) is estimated by \( n/m \). Similar to the proof of Lemma 4.2 in the Appendix and the standard methods in nonparametric density estimation, it can be shown that,

\[
\hat{f}(\theta_0) = \frac{1}{m} \sum_{i=1}^{m} \alpha_{1m}(x_i; x_{ij}, \theta_{m,n}, \Delta), \quad \hat{g}(\theta_0 + \Delta) = \frac{1}{n} \sum_{j=1}^{n} \alpha_{2n}(y_j; y_{ji}, \theta_{m,n}, \Delta)
\]

are consistent estimators of \( \hat{f}(\theta_0) \) and \( \hat{g}(\theta_0 + \Delta) \), respectively. In this way, we can get a consistent estimator \( \hat{a}_0(\Delta) \) of \( a_0(\Delta) \).

Let \( t_\alpha \) satisfy \( P(\chi_1^2 \leq t_\alpha) = 1 - \alpha \). It follows from Theorem 2.1, that an EL based confidence interval on \( \Delta \) with asymptotically correct coverage probability \( 1 - \alpha \)

can be constructed as \( \Delta : -2\hat{a}_0(\Delta)R(\Delta, \theta_{m,n}) \leq t_\alpha \). We also notice that the result can apply to the complete data settings. In complete data situation, \( P_1 = P_2 = 1 \). Thus we can see that the asymptotic distribution of EL statistic is found to be a \( \chi_1^2 \) distribution. The EL based confidence interval for \( \Delta \) in complete data case is thus constructed as

\[ \Delta : -2R(\Delta, \theta_{m,n}) \leq t_\alpha \].

**APPENDIX**

To prove the main result Theorem 2.1, we need the following lemmas.

**Lemma 4.1**\(^{(9)}\) Let \( U_n, V_n \) be two sequences of random variables and \( B_n \) be a \( \sigma - \) algebra. Assume that:

1. There exists \( \sigma_{2n}^2 > 0 \) such that \( \sigma_{2n}^{-1} V_n \rightarrow N(0, 1) \) as \( n \rightarrow \infty \), \( V_n \) is \( B_n \) measurable.
2. \( E(U_n | B_n) = 0, Var(U_n | B_n) = \sigma_{2n}^2 \) such that

\[
\sup_t \left| P(\sigma_{2n}^{-1} U_n \leq t | B_n) - \Phi(t) \right| = o_p(1)
\]

where \( \Phi(\cdot) \) is the distribution function of the standard normal random variable.

3. \( \gamma_n^2 = \frac{\sigma_{2n}^2}{\sigma_{2n}^2} = \gamma^2 + o_p(1) \), then as \( n \rightarrow \infty \)

\[
\frac{U_n + V_n}{\sqrt{\sigma_{2n}^2 + \sigma_{2n}^2}} \rightarrow d N(0, 1).
\]
Lemma 4.2 Under the conditions of Theorem 2.1, as \( mn \to \infty \),
\[
\frac{1}{m} \sum_{i=1}^{m} \omega_{im}(x_i, y_{ij}, \theta, \Delta) \overset{d}{\to} N(0, \sigma^2_1), \quad \frac{1}{n} \sum_{j=1}^{n} \omega_{jn}(y_j, y_{ij}, \theta, \Delta) \overset{d}{\to} N(0, \sigma^2_2)
\]
and
\[
\frac{1}{m} \sum_{i=1}^{m} \omega_{im}^2(x_i, x_{ij}, \theta, \Delta) = q(1 - q) + o_p(1), \quad \frac{1}{n} \sum_{j=1}^{n} \omega_{jn}^2(y_j, y_{ij}, \theta, \Delta) = q(1 - q) + o_p(1).
\]
Where \( \sigma^2_1 = \left(1 - \frac{p_i}{k} + p_i^{-1}\right)q(1 - q), \quad \sigma^2_2 = \left(1 - \frac{p_i^2}{k} + p_i^2^{-1}\right)q(1 - q). \)

Proof of Lemma 4.2 Let
\[
\omega_i(x_i, \theta, \Delta) = \int_{-\infty}^{a} K_i(u)du - q, \quad \omega_k(x_i, \theta, \Delta) = \frac{1}{k} \sum_{i=1}^{m} \omega_i(x_i, \theta, \Delta),
\]
\[
\overline{\omega}_i = \frac{1}{r_x} \sum_{x \in r_x} \omega_i(x, \theta_0, \Delta), \quad B_m = \sigma((\delta, x), i = 1, 2 \ldots, m)
\]
Then under the assumption of MCAR, we have
\[
E[\sum_{i \in r_x} \omega_i(x_i, \theta_0, \Delta) | B_m] = \frac{1}{r_x} \sum_{x \in r_x} \omega_i(x, \theta_0, \Delta) = \overline{\omega}_i,
\]
\[
Var[\omega_i(x_i, \theta_0, \Delta) | B_m] = E[\omega_i(x_i, \theta_0, \Delta) - E\omega_i(x_i, \theta_0, \Delta) | B_m]^2
\]
\[
= \frac{1}{r_x} \sum_{x \in r_x} [\omega_i(x, \theta_0, \Delta) - E\omega_i(x, \theta_0, \Delta)]^2 = Var\omega_i(x, \theta_0, \Delta),
\]
So
\[
E[\sum_{i \in r_x} \omega_k(x_i, \theta_0, \Delta) | B_m] = E[\frac{1}{k} \sum_{i=1}^{k} \omega_i(x_i, \theta_0, \Delta) | B_m] = \frac{1}{k} \sum_{i=1}^{k} E[\omega_i(x_i, \theta_0, \Delta) | B_m] = \overline{\omega}_k,
\]
and
\[
Var[\omega_k(x_i, \theta_0, \Delta) | B_m] = Var[\frac{1}{k} \sum_{i=1}^{k} \omega_i(x_i, \theta_0, \Delta) | B_m] = \frac{1}{k} Var\omega_i(x, \theta_0, \Delta).
\]
It follows that
\[
\frac{1}{\sqrt{m}} \sum_{i=1}^{m} \omega_{im}(x_i, y_{ij}, \theta, \Delta) = \sqrt{m} \overline{\omega}_d + \frac{1}{\sqrt{m}} \sum_{i \in r_x} [\omega_i(x_i, \theta, \Delta) - E\omega_i(x_i, \theta, \Delta) | B_m] \square U_m + V_m, \quad V_m \text{ is } B_m \text{ measurable, and}
\]
\[
V_m = \sqrt{m} \frac{1}{r_x} \sum_{x \in r_x} [\omega_i(x, \theta_0, \Delta) - E\omega_i(x, \theta_0, \Delta)] + \sqrt{m} E\omega_i(x, \theta_0, \Delta).
\]
From [1] it can be shown that
\[
E\omega_i(x_i, \theta_0, \Delta) = O(a^\nu), \quad Var\omega_i(x_i, \theta_0, \Delta) = q(1 - q) + o_p(1),
\]
Thus from Assumptions (2.3) and (2.5) we have \( \sqrt{m} E\omega_i(x_i, \theta_0, \Delta) = o(1) \). Combining with the MCAR assumption and
the central Limit Theorem (CLT), we have

\[ V_m \xrightarrow{d} N(0, p_1^{-1} q(1-q)), \]

And \( E(U_m | B_m) = 0, \)

\[ Var(U_m | B_m) = \frac{m}{m} Var(\omega_\lambda(x_i, \theta_0, \Delta) | B_m) = \frac{m}{m} \frac{1}{k} Var(\omega_\lambda(x_i, \theta_0, \Delta) = \frac{1}{k} p_1 q(1-q) + o_p(1). \]

Thus

\[ U_m \xrightarrow{d} N(0, \frac{1}{k} p_1 q(1-q)). \]

From Berry-Esseen’s Central Limit Theorem for independent random variables, we have

\[ \sup_t | P(\sigma_{2m}^{-1} U_m \leq t | B_m) - \Phi(t) | = o_p(1), \quad \text{where} \quad \sigma_{2m}^2 = \frac{1}{k} p_1 q(1-q). \]

Hence, from Lemma 4.1, we have

\[ \frac{1}{m} \sum_{i=1}^{m} \omega_{\lambda m}(x_i; x_i, \theta_0, \Delta) \xrightarrow{d} N(0, \sigma_1^2) \]

On the other hand, denote the conditional probability given \( B_m \) as \( p^* \). Then by the law of large numbers and MCAR assumption,

\[ \frac{1}{m} \sum_{i=x}^{m} \omega_{\lambda k}(x_i, \theta_0, \Delta) = E(\omega_{\lambda k}(x_i, \theta_0, \Delta) | B_m) + o_p(1) = E\left[ \frac{1}{k} \sum_{i=1}^{k} \omega_{\lambda k}(x_i, \theta_0, \Delta) | B_m \right]^2 + o_p(1) \]

\[ = \frac{1}{k} E\left[ \sum_{i=1}^{k} (\omega_{\lambda k}(x_i, \theta_0, \Delta) - E(\omega_{\lambda k}(x_i, \theta_0, \Delta) | B_m) \right]^2 + o_p(1) \]

\[ = \frac{1}{k} E\left[ \sum_{i=1}^{k} (\omega_{\lambda k}(x_i, \theta_0, \Delta) - E(\omega_{\lambda k}(x_i, \theta_0, \Delta) | B_m) \right]^2 + E(\omega_{\lambda k}(x_i, \theta_0, \Delta) | B_m)^2 + o_p(1) \]

\[ = \frac{1}{k} Var(\omega_{\lambda k}(x_i, \theta_0, \Delta) | B_m) + [\omega_{\lambda k}]^2 + o_p(1) = \frac{1}{k} Var + o_p(1) = \frac{1}{k} q(1-q) + o_p(1) \]

It follows that

\[ \frac{1}{m} \sum_{i=x}^{m} \omega_{\lambda m}(x_i; x_i, \theta_0, \Delta) \xrightarrow{d} N(0, \sigma_1^2) \]

\[ \frac{1}{m} \sum_{i=x}^{m} \omega_{\lambda m}(x_i; x_i, \theta_0, \Delta) + \frac{1}{m} \sum_{i=x}^{m} \omega_{\lambda m}(x_i; x_i, \theta_0, \Delta) \]

\[ = \frac{1}{k} E\omega_\lambda^2(x_i, \theta_0, \Delta) + \frac{1}{k} P_1 \omega_\lambda^2(x_i, \theta_0, \Delta) + o_p(1) = \frac{1}{k} - \frac{1}{k} + \frac{k}{k} E\omega_\lambda^2(x_i, \theta_0, \Delta) + o_p(1) \]

\[ = \frac{1}{k} - \frac{1}{k} + \frac{k}{k} q(1-q) + o_p(1) \]

The rest of Lemma 4.2 can be proved similarly. So the proof of Lemma 4.2 is complete.

The other hand, similar to [1],[8],[10] we have the lemma 4.3 and Lemma 4.4 as follows:

**Lemma 4.3** Suppose that \( 1/3 < \eta < 1/2 \) and the conditions of Theorem 2.1 are satisfied. Then, as \( m, n \to \infty \),

\[ \lambda_1(\theta) = O_p(n^{-\eta}), \quad \lambda_2(\theta) = O_p(n^{-\eta}) \]

uniformly about \( \theta \in \{ \theta : | \theta - \theta_0 | \leq cn^{-\eta} \} \), where \( c \) is some positive constant.

**Lemma 4.4** Suppose that \( 1/3 < \eta < 1/2 \) and the conditions of Theorem 2.1 are satisfied. Then with probability tending to 1,
there exists a root $\theta_{m,n}$ of equation (2.3), such that, as $m,n \to \infty$, 
\[
|\theta_{m,n} - \theta_0| = O_p(n^{-\eta})
\]
and $R(\Delta, \theta)$ attains its local maximum value at $\theta_{m,n}$.

**Lemma 4.5** Suppose that the conditions of Theorem 2.1 are satisfied, and $\theta_{m,n}$ is as that in Lemma 4.4, then, as $m,n \to \infty$
\[
\sqrt{m(\theta_{m,n} - \theta_0)} \to N(0, \frac{1}{c_0} \left( \frac{1-p_2+kp_2}{k^2} f^2(\theta_0) \sigma_1^2 + \frac{h(1-p_1+kp_1)^2 g^2(\theta_0 + \Delta)}{k^2} \sigma_2^2 \right))
\]
\[
\lambda_2(\theta_{m,n}) = -\frac{h g(\theta_0 + \Delta)}{f(\theta_0)} \lambda_2(\theta_{m,n}), \quad \sqrt{m\lambda_2(\theta_{m,n})} \to N(0, \sigma^2)
\]

Where
\[
\sigma^2 = (q(1-q))^{-1} f^2(\theta_0) \frac{g^2(\theta_0 + \Delta)\sigma_1^2 + h f^2(\theta_0)\sigma_2^2}{c_0^2},
\]
\[
\sigma_1^2, \sigma_2^2, c_0 \text{ are defined in Lemma 4.2 and Theorem 2.1.}
\]

**Proof of Lemma 4.5** Let $\lambda_1 = \lambda_1(\theta)$, $\lambda_{E_1} = \lambda_{E_1}(\theta_{m,n})$, $\lambda_2 = \lambda_2(\theta)$, $\lambda_{E_2} = \lambda_{E_2}(\theta_{m,n})$, and
\[
Q_{1,m,n}(\theta, \lambda_1, \lambda_2) = \frac{1}{m} \sum_{i=1}^{m} \omega_{m}(x_i; x_j, \theta, \Delta),
\]
\[
Q_{2,m,n}(\theta, \lambda_1, \lambda_2) = \frac{1}{n} \sum_{j=1}^{n} \omega_{n}(y_j; y_{ij}, \theta, \Delta),
\]
\[
Q_{3,m,n}(\theta, \lambda_1, \lambda_2) = \frac{\lambda_1}{ma} \sum_{i=1}^{m} \frac{\delta_{a}K_{1}(\theta-x_i)}{1+\lambda_1(\theta)\omega_{m}(x_i; x_j, \theta, \Delta)} + \frac{1-\delta_{a}}{a} \sum_{k=1}^{k} \frac{K_{1}(\theta-x_i)}{a}
\]
\[
+ \frac{\lambda_2}{mb} \sum_{j=1}^{b} \frac{\delta_{b}K_{2}(\theta+\Delta-y_{ij})}{1+\lambda_2(\theta)\omega_{n}(y_j; y_{ij}, \theta, \Delta)} + \frac{1-\delta_{b}}{b} \sum_{k=1}^{b} \frac{K_{2}(\theta+\Delta-y_{ij})}{b}
\]

From Lemma 4.4, we have $Q_{i,m,n}(\theta, \lambda_1, \lambda_2) = 0$, $i = 1, 2, 3$. From Taylar expansion, Lemma 4.3 and 4.4, we have
\[
0 = Q_{i,m,n}(\theta, \lambda_1, \lambda_2) = Q_{i,m,n}(\theta, 0, 0) + \frac{\partial Q_{i,m,n}(\theta, 0, 0)}{\partial \theta} (\theta_{m,n} - \theta_0)
\]
\[
+ \frac{\partial Q_{i,m,n}(\theta, 0, 0)}{\partial \lambda_1} \lambda_{E_1} + \frac{\partial Q_{i,m,n}(\theta, 0, 0)}{\partial \lambda_2} \lambda_{E_2} + o_p(\xi_n), \quad i = 1, 2, 3
\]

Where $\xi_n = |\theta_{m,n} - \theta_0| + |\lambda_{E_1}| + |\lambda_{E_2}|$. Hence, as $i = 1, 2, 3$ we have
\[
-Q_{i,m,n}(\theta_0, 0, 0) = \frac{\partial Q_{i,m,n}(\theta_0, 0, 0)}{\partial \theta} (\theta_{m,n} - \theta_0) + \frac{\partial Q_{i,m,n}(\theta_0, 0, 0)}{\partial \lambda_1} \lambda_{E_1} + \frac{\partial Q_{i,m,n}(\theta_0, 0, 0)}{\partial \lambda_2} \lambda_{E_2} + o_p(\xi_n)
\]
It can be shown,
similar to the proof Lemma 4.2, that

\[
\frac{\partial Q_{i,m,n}(\theta_0, 0, 0)}{\partial \theta} = \frac{1}{m} \sum_{i=1}^{m} \alpha_{i,m}(x_i; x_j, \theta, \Delta) = f(\theta_0) + o_p(1)
\]

At the same time we have

\[
\frac{\partial Q_{1,m,n}(\theta_0, 0, 0)}{\partial \lambda_2} = 0, \quad \frac{\partial Q_{2,m,n}(\theta_0, 0, 0)}{\partial \lambda_4} = 0, \quad \frac{\partial Q_{3,m,n}(\theta_0, 0, 0)}{\partial \theta} = 0
\]

\[
\frac{\partial Q_{2,m,n}(\theta_0, 0, 0)}{\partial \theta} = g(\theta_0 + \Delta) + o_p(1), \quad \frac{\partial Q_{2,m,n}(\theta_0, 0, 0)}{\partial \lambda_2} = -\frac{(1 - p_1 + k p_1)}{k} q(1 - q) + o_p(1)
\]

\[
\frac{\partial Q_{3,m,n}(\theta_0, 0, 0)}{\partial \lambda_2} = h g(\theta_0 + \Delta) + o_p(1)
\]

Thus

\[
\left( \begin{array}{c}
\theta_{m,n} - \theta_0 \\
\lambda_{1E_i} \\
\lambda_{2E_i}
\end{array} \right) = s^{-1} \left( \begin{array}{c}
-Q_{1,m,n}(\theta_0, 0, 0) \\
-Q_{2,m,n}(\theta_0, 0, 0) \\
0
\end{array} \right) + o_p(\xi_n)
\]

Where

\[
s^{-1} = \frac{1}{(1 - q)c_0} \left( \begin{array}{ccc}
(1 - p_1 + k p_1) q(1 - q) f(\theta_0) & (1 - p_1 + k p_1) h q(1 - q) g(\theta_0 + \Delta) & (1 - p_1 + k p_1) kq(1 - q) f(\theta_0) \\
- h f(\theta_0) & h f(\theta_0) g(\theta_0 + \Delta) & (1 - p_1 + k p_1) k g(\theta_0 + \Delta) \\
(1 - p_1 + k p_1) k q(1 - q) f(\theta_0) & (1 - p_1 + k p_1) k q(1 - q) g(\theta_0 + \Delta) & (1 - p_1 + k p_1) k q(1 - q) g(\theta_0 + \Delta)
\end{array} \right)
\]

Combining with \( \sqrt{n} Q_{i,m,n}(\theta, 0, 0) = o_p(1), i = 1, 2 \). We have \( \xi_n = O_p(n^{-1/2}) \). It follows that

\[
\theta_{m,n} - \theta_0 = -\frac{1}{c_0} \left( \begin{array}{c}
(1 - p_1 + k p_1) k \frac{f(\theta_0)}{k} \\
-h f(\theta_0) g(\theta_0 + \Delta) \\
(1 - p_1 + k p_1) k g(\theta_0 + \Delta)
\end{array} \right) + o_p(n^{-1/2})
\]

\[
\lambda_{1E_i} = \frac{1}{q(1 - q)c_0} \left( \begin{array}{c}
(1 - p_1 + k p_1) q f(\theta_0) Q_{1,m,n}(\theta_0, 0, 0) + \frac{h f(\theta_0) g f(\theta_0) Q_{2,m,n}(\theta_0, 0, 0) + o_p(n^{-1/2})}
\end{array} \right)
\]

\[
\lambda_{2E_i} = \frac{1}{q(1 - q)c_0} \left( \begin{array}{c}
(1 - p_1 + k p_1) q f(\theta_0) Q_{1,m,n}(\theta_0, 0, 0) + \frac{h f(\theta_0) g(\theta_0 + \Delta) Q_{2,m,n}(\theta_0, 0, 0) + o_p(n^{-1/2})}
\end{array} \right)
\]

then, it is easy obtain

\[
\lambda_{1E_i} = \frac{1}{q(1 - q)c_0} \left( \begin{array}{c}
(1 - p_1 + k p_1) q f(\theta_0) Q_{1,m,n}(\theta_0, 0, 0) + \frac{h f(\theta_0) g(\theta_0 + \Delta) Q_{2,m,n}(\theta_0, 0, 0) + o_p(n^{-1/2})}
\end{array} \right)
\]

From Lemma 4.2, we have
Thus we have Lemma 4.5.

**Proof of Theorem 2.1** Similar to the proof of Theorem 2.1 in [5].

\[
-2R(\Delta, \theta_{m,n}) = m \lambda_1^2(\theta_{m,n}) \cdot \frac{1}{m} \sum_{i=1}^{m} \omega^{2}_{1m}(x_i; x_i', \theta_i, \Delta) + n \lambda_2^2(\theta_{m,n}) \cdot \frac{1}{n} \sum_{j=1}^{n} \omega^{2}_{2n}(y_j; y_j', \theta_j, \Delta) + o_p(1)
\]

\[
= a_0(\Delta) \left( \sqrt{\frac{m}{\sigma}} \hat{\lambda}_{\Delta} \right)
\]

Combining with Lemma 4.2 and Lemma 4.5, we have Theorem 2.1.

**REFERENCES**