Further properties on strongly generalized star semi-continuous mappings

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Abstract: The aim of this paper is to introduce and study the class of strongly generalized star semi-closed sets which is weaker than semi-closed sets (Crossly and Hildebrand, 1971) and stronger than both strongly generalized semi-closed sets (El-Maghrabi and Nasef, 2008) and semi generalized-closed sets (Bhattacharya and Lahiri, 1987). Also, through this paper some concepts such as: strongly generalized star semi-continuous, strongly generalized star semi-closed and strongly generalized star semi-homeomorphism maps are discussed and investigated via a strongly generalized star semi-closed set.

(1991) AMS Math. Subject Classification: 54 A05; 54 D10

Keywords and Phrases: strongly generalized star semi-closed sets, strongly generalized star semi-continuous, strongly generalized star semi-irresolute, strongly generalized star semi-closed and strongly generalized star semi-homeomorphism mappings.

INTRODUCTION

In 1970, Levine [15] introduced the concept of generalized closed (briefly, g-closed) sets of a topological space. Bhattacharya and Lahiri [4] defined and studied the notion of sg-closed sets. In 1990, Arya and Nour [2] introduced the concept of gs-closed sets. Veera Kumar [21] defined and studied the notion of $g^*$-closed sets. The notion of $g^*$-closed sets was defined by El-Maghrabi and Nasef [12]. The purpose of the present paper is to define and investigate the concept of strongly generalized star semi-closed sets. Some notions are introduced and investigated via a strongly generalized star semi-closed set such as: strongly generalized star semi-continuity, strongly generalized star semi-irresoluteness, strongly generalized star semi-closed and strongly generalized star semi-homeomorphism maps.

PRELIMINARIES

Throughout this paper, spaces always mean topological spaces on which no separation axiom is assumed unless explicitly stated. Let $X$ be a space and $A$ be a subset of $X$. The closure of $A$ and the interior of $A$ are denoted by $\text{cl}(A)$ and $\text{int}(A)$ respectively. A subset $A$ of $X$ is said to be regular-open [19] (resp. semi-open [14], pre-open [17], Q-set [13]) if $\text{int}(\text{cl}(A)) \subseteq A$ (resp. $A \subseteq \text{cl}(\text{int}(A))$, $\text{int}(\text{cl}(A)) \subseteq \text{cl}(\text{int}(A))$). A subset $A$ of $X$ is said to be semi-open or, equivalently, if $\text{int}(\text{cl}(A)) \subseteq A$ [8]. The family of all semi-open (resp. semi-closed) sets will be denoted by $S_{0}(X,\tau)$ (resp. $S_{c}(X,\tau)$). The intersection (resp. the union) of all semi-closed (resp. semi-open) sets containing (resp. contained in) $A$ is called the semi-closure (resp. the semi-interior) of $A$ and will be denoted by $S\cap A$ (resp. $S\cap A$).

Definition 2.1. A subset of a space $(X,\tau)$ is called:

1- a generalized closed (briefly, g-closed) [15] set if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,
2- a semi generalized-closed (briefly, sg-closed) [4] set if $S\cap A \subseteq U$ whenever $A \subseteq U$ and $U$ is semi-open,
3- a generalized semi-closed (briefly, gs-closed)[2] set if $S\cap A \subseteq U$ whenever $A \subseteq U$ and $U$ is open,
4- a strongly generalized semi-closed (briefly, $g^*$-closed) [12] set if $S\cap A \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open,
5- a $g^*$-closed [21] set if $cI(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open.

**Remark 2.1.** The complement of g-closed (resp. sg-closed, gs-closed, $g^*s$-closed, $g^*$-closed) is called g-open (resp. sg-open, gs-open, $g^*s$ -open, $g^*$ -open).

**Definition 2.2.** A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is called g-continuous [3] (resp. sg-continuous [20], gs-continuous [11], $g^*$-continuous [21]) if $f^{-1}(V)$ is g-closed (resp. sg-closed, gs-closed, $g^*$ -closed) in $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.

**Definition 2.3.** A mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

(i) g-closed [16] (resp. sg-closed [11], gs-closed [11]) if $f(V)$ is g-closed (resp. sg-closed, gs-closed) in $(Y, \sigma)$ for every closed set $V$ of $(X, \tau)$.

(ii) g-open [16] (resp. sg-open [11], gs-open [11]) if $f(V)$ is g-open (resp. sg-open, gs-open) in $(Y, \sigma)$ for every open set $V$ of $(X, \tau)$.

**Definition 2.4.** A bijective mapping $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

(i) semi-homeomorphism (B) [5] if $f$ is semi-continuous and semi-open,

(ii) semi-generalized-homeomorphism [10] (briefly, sg-homeomorphism), if $f$ is sg-continuous and sg-open,

(iii) generalized semi-homeomorphism [10] (briefly, gs-homeomorphism), if $f$ is gs-continuous and gs-open.

**Lemma 2.1** [7,8,9]. If $A$ and $B$ are two subsets of $X$, then the following statements are hold:

(i) $s-cl(A)$ (resp. $s-int(A)$) is semi- closed (resp. semi-open),

(ii) $A$ is semi- closed (resp. semi-open) iff $s-cl(A)$ (resp. $s-int(A)$),

(iii) $s-cl(X-A) = X - s-int(A)$ and $s-int(X-A) = X - s-cl(A)$,

(iv) $A \subseteq s-cl(A) \cup s-int(A)$,

(v) $s-cl(s-cl(A)) = s-cl(A)$.

**Corollary 2.1** [1]. Let $A$ be a subset of a space $(X, \tau)$. Then $s-cl(A) \cup s-int(cl(A))$.

3. More on strongly $g^*s$-closed sets.

**Definition 3.1** A subset $A$ of a space $X$ is called a strongly generalized star semi-closed (briefly, strongly $g^*s$-closed) set, if $U(A)cl \subseteq U$ whenever $UA \subseteq U$ and $U$ is gs-open in $(X, \tau)$.

A subset $B$ of a space $(X, \tau)$ is called a strongly generalized star semi-open (briefly, strongly $g^*s$-open) set, if $X-B$ is strongly generalized star semi-closed in $(X, \tau)$.

**Remark 3.1.** The concepts of g-closed (resp. $g^*$-closed) and strongly $g^*s$-closed sets are independent.

**Example 3.1.** If $X=\{a,b,c,d\}$ with two topologies $\tau_1, \tau_2$ on $X$ such that: $\tau_1=\{X, \varnothing, \{a\}, \{a,b\}\}, \tau_2=\{X, \varnothing, \{a\}, \{b,c\}, \{a,b,c\}\}$, then:

(1) a subset $A=\{b\}$ of $X$ on $\tau_1$ is strongly $g^*s$-closed but not g-closed and a subset $B=\{a,b,d\}$ of $X$ on $\tau_1$ is g-closed but not strongly $g^*s$-closed.

(2) a subset $C=\{a\}$ of $X$ on $\tau_2$ is strongly $g^*s$-closed but not $g^*$-closed and a subset $D=\{b,d\}$ of $X$ on $\tau_2$ is $g^*$-closed but not strongly $g^*s$-closed.

**Remark 3.2.** By Definition 3.1 and Remark 3.1, we obtain the following diagram.
However, the converses are not true in [2, 9, 12, 21] and by the following examples.

**Example 3.2.** If $X = \{a, b, c, d\}$ with topologies $\tau_1,$ $\tau_2$ on $X$ such that:

$\tau_1 = \{X, \emptyset, \{c,d\}\},$ $\tau_2 = \{X, \emptyset, \{c\}, \{c,b\}, \{b,c,d\}\},$ then a subset $A = \{a, b, c\}$ of $X$ on $\tau_1$ is strongly $g^s$-closed but not semi-closed. While, a subset $B = \{a, c\}$ of $X$ on $\tau_2$ is $g^s$-closed but not strongly $g^s$-closed.

**Example 3.3.** Let $X = \{a, b, c\}$ with topologies $\tau_1, \tau_2$ on $X$ such that:

$\tau_1 = \{X, \emptyset, \{a,b\}, \{c\}\},$ $\tau_2 = \{X, \emptyset, \{a\}, \{a,b\}\}.$ Then, a subset $C = \{a\}$ of $X$ on $\tau_1$ is sg-closed but not strongly $g^s$-closed. But a subset $D = \{a, c\}$ of $X$ on $\tau_2$ is $g^s$-closed but not strongly $g^s$-closed.

**Remark 3.3.** The union of two strongly $g^s$-closed sets need not be strongly $g^s$-closed. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}\}.$ Then, the subsets $A = \{a\}$ and $B = \{b\}$ are strongly $g^s$-closed but their union is not strongly $g^s$-closed.

**Theorem 3.1.** A subset $A$ of a space $(X, \tau)$ is strongly $g^s$-closed if and only if every $g$-open set $G$ containing $A$, there exists a semi-closed set $F$ such that $G \subseteq A \subseteq F \subseteq G.$

**Proof.** Necessity. Let $A$ be a strongly $g^s$-closed set, $A \subseteq G$ and $G$ be $g$-open. Then $s - \text{cl}(A) \subseteq G.$ Set, $s - \text{cl}(A) = F.$ Hence, there exists a semi-closed set $F$ such that $A \subseteq F \subseteq G.$

Sufficiency. Assume that $A \subseteq G$ and $G$ is a $g$-open set of $X.$ Then by hypothesis, there exists a semi-closed set $F$ such that $A \subseteq F \subseteq G,$ therefore, $s - \text{cl}(A) \subseteq G.$ So, $A$ is strongly $g^s$-closed.

**Theorem 3.2.** Let $A$ be a strongly $g^s$-closed set of $X.$ Then $(s - \text{cl}(A)) - A$ does not contain any non empty $g$-closed set.

**Proof.** Let $F$ be a $g$-closed set such that $F \subseteq (s - \text{cl}(A)) - A.$ Then $F \subseteq X - A$ this implies that $A \subseteq X - F.$ Since, $A$ is strongly $g^s$-closed and $X - F$ is $g$-open, then $s - \text{cl}(A) \subseteq X - F,$ that is $F \subseteq X - (s - \text{cl}(A)),$ hence $F \subseteq s - \text{cl}(A) \cap (X - (s - \text{cl}(A))) \emptyset.$ This shows that $F \emptyset.$

The converse of the above theorem may not be true as is shown by the following example.

**Example 3.4.** In Example 3.1, if $A = \{a, b, d\}$ is a subset of $X$ on a topology $\tau_2$, then $(s - \text{cl}(A)) - A \emptyset \{c\}$ does not contain any non empty $g$-closed set.

**Corollary 3.1.** Let $A$ be a strongly $g^s$-closed set of $X.$ Then $(s - \text{cl}(A)) - A$ does not contain any non empty $gs$-closed set.

**Proof.** Obvious.

**Corollary 3.2.** Let $A$ be a strongly $g^s$-closed set. Then $A$ is semi-closed if and only if $(s - \text{cl}(A)) - A$ is $gs$-closed.

**Proof.** Necessity. Assume that $A$ is strongly $g^s$-closed and semi-closed sets. Then $s - \text{cl}(A)$ $A$ and hence $(s - \text{cl}(A)) - A \emptyset$ which is $gs$-closed.

Sufficiency. Suppose that $s - \text{cl}(A) - A$ is $gs$-closed and $A$ is strongly $g^s$-closed. Then by Corollary 3.1, $s - \text{cl}(A) - A$ does not contain any non empty $gs$-closed subset of $X.$ Hence $A$ is semi-closed.

**Theorem 3.3.** For each $x \in X,$ then $\{x\}$ is $gs$-closed or its complement $X - \{x\}$ is strongly $g^s$-closed.
Proof. Suppose that \( \{x\} \) is not gs- closed. Then its complement is not gs- open. Since, \( X \) is the only gs- open set containing \( X-\{x\} \), that is, \( s-scl(X-\{x\}) \subseteq X \) holds. This implies that \( X-\{x\} \) is strongly g*s- closed.

**Proposition 3.1.** If \( A \) is a strongly g*s -closed set and \( A \subseteq B \subseteq s-scl(A) \), then \( B \) is strongly g*s- closed.

**Proof.** Let \( B \subseteq U \) and \( U \) be a gs- open set of \( X \). Then \( A \subseteq U \). Since \( A \) is strongly g*s - closed, hence \( s-scl(A) \subseteq U \), but \( B \subseteq s-scl(A) \). Then \( s-scl(B) \subseteq U \). Hence, \( B \) is strongly g*s- closed.

**Proposition 3.2.** If \( (X, \tau) \) is a topology space and \( A \subseteq X \), then \( A \) is semi – closed, if one of the following two cases hold:

1. If \( A \) is strongly g*s-closed and gs-open.
2. If \( A \) is strongly g*s-closed and open.

**Theorem 3.4.** Let \( A \) be a subset of a space \( X \), the following are equivalent:

(i) \( A \) is regular – open,
(ii) \( A \) is open and strongly g*s-closed.

**Proof.** (i) \( \Rightarrow \) (ii). Let \( U \) be a gs-open set containing \( A \) and \( A \) be a regular-open set. Then, \( A \cup int(cl(A)) \subseteq U \). So, \( s-cl(A) \subseteq U \) and therefore \( A \) is strongly g*s-closed.

(ii) \( \Rightarrow \) (i). Hence by Theorem 3.4, \( A \) is regular-open. Since, every regular-open set is open, then \( A \) is a Q-set, hence \( A \) is closed. Therefore, \( A \) is clopen.

**Theorem 3.5.** If \( A \) is a subset of a space \( X \), the following are equivalent:

(i) \( A \) is clopen,
(ii) \( A \) is open, a Q-set and strongly g*s-closed.

**Proof.** (i) \( \Rightarrow \) (ii). Since \( A \) is clopen, hence \( A \) is both open and a Q-set. Let \( U \) be a gs-open set containing \( A \). Then, \( A \cup int(cl(A)) \subseteq U \) and so \( s-cl(A) \subseteq U \). Hence, \( A \) is strongly g*s-closed.

(ii) \( \Rightarrow \) (i). Hence by Theorem 3.4, \( A \) is regular-open. Since, every regular-open set is open, then \( A \) is a Q-set, hence \( A \) is closed. Therefore, \( A \) is clopen.

**Theorem 3.6.** For a subset \( A \) of a space \( X \), the following statements are equivalent:

(i) \( A \) is strongly g*s - open,
(ii) For each gs-closed set \( F \subseteq X \) contained in \( A \), \( F \subseteq s-int(A) \),
(iii) For each gs-closed set \( F \subseteq X \) contained in \( A \), there exists a semi-open set \( G \subseteq X \) such that \( F \subseteq G \subseteq A \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( F \subseteq A \) and \( F \) be a gs- closed set. Then \( X-A \subseteq X-F \) which is gs-open. Hence, \( s-cl(X-A) \subseteq X-F \). Therefore by Lemma 2.1, (iii), \( F \subseteq s-int(A) \).

(ii) \( \Rightarrow \) (iii). Let \( F \subseteq A \) and \( F \) be a gs-closed set. Then by hypothesis, \( F \subseteq s-int(A) \). Set \( s-int(A) \subseteq G \), hence \( F \subseteq G \subseteq A \).

(iii) \( \Rightarrow \) (i). Let \( X-A \subseteq U \) and \( U \) be a gs-open set. Then \( X-U \subseteq A \) and by hypothesis, there exists a semi-open set \( G \) such that \( X-U \subseteq G \subseteq A \), that is, \( X-A \subseteq X-G \subseteq U \). Therefore, by Theorem 3.1, \( X-A \) is strongly g*s- closed. Hence, \( A \) is strongly g*s-open.

**Lemma 3.1.** Let \( A \subseteq X \) be a strongly g*s -closed set. Then \( s-cl(A)-A \) is strongly g*s - open.

**Proof.** Let \( F \) be a gs- closed set such that \( F \subseteq (s-cl(A))-A \). Since \( A \) is strongly g*s-closed, then by Corollary 3.1, \( F \not\subseteq \emptyset \). Therefore, \( \emptyset \subseteq s-int(s-cl(A)-A) \). Hence, by Theorem 3.6, \( s-cl(A)-A \) is strongly g*s- open.

4. **Strongly g*s-continuous mappings.**

**Definition 4.1.** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called a strongly generalized star semi-continuous (briefly, strongly g*s-continuous) mapping if the inverse image of each closed set in \( Y \) is strongly g*s-closed in \( X \).

**Definition 4.2.** A mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) is called strongly generalized star semi-irresolute (briefly, strongly g*s- irresolute) if, \( f^{-1}(U) \) is strongly g*s-closed in \((X, \tau)\), for every strongly g*s-closed set \( U \) of \((Y, \sigma)\).

**Lemma 4.1.** (1) Every semi- continuous mapping is strongly g*s-continuous.
Every strongly $g^*s$-continuous mapping is sg-continuous (resp. gs-continuous).

**Remark 4.1.** The concept of strongly $g^*s$-continuous and g-continuous (resp. $g^s$-continuous) mappings are independent, as is shown by the following examples.

**Example 4.1.** Let $X = \{a,b,c,d\}$, $Y = \{a,b,c\}$ with two topologies $\tau_X = \{X, \varphi, \{a\}\}$, $\tau_Y = \{Y, \varphi, \{a\}\}$ and a mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is defined by $f(a) = b$, $f(b) = c$ and $f(c) = f(d) = a$, then $f$ is $g$-continuous but not strongly $g^*s$-continuous.

**Example 4.2.** If $X = \{a,b,c\}$ with topologies.

(i) $\tau_X = \{X, \varphi, \{a\}, \{b\}, \{a,b\}\}$, $\tau_Y = \{Y, \varphi, \{a\}, \{b,c\}\}$, then a mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ which is defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ is strongly $g^*s$-continuous but not $g^*$-continuous.

(ii) $\tau_X = \{X, \varphi, \{b\}, \{a,c\}\}$, $\tau_Y = \{Y, \varphi, \{b,c\}\}$, then a mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ which is defined by $f(a) = f(b) = a$ and $f(c) = b$ is $g^*$-continuous but not strongly $g^*s$-continuous.

**Remark 4.2.** By Lemma 4.1 and Remark 4.1, we have the following diagram.

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continuity  \downarrow\  g^* - continuity \downarrow\  g^s - continuity \downarrow\  gs - continuity
semi-continuity  \downarrow\  strongly g^s - continuity  \downarrow\  sg - continuity
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The converses of this implication is not true in [6,11,14,21] and by the following examples.

**Example 4.3.** Let $X = \{a,b,c\}$, $Y = \{a,b,c\}$, and a mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be defined by $f(a) = f(c) = a$ and $f(b) = b$. Then $f$ is strongly $g^*s$-continuous but not semi-continuous.

**Example 4.4.** If $X = \{a,b,c\}$, $\tau_X = \{X, \varphi, \{a,b\}, \{b,c\}\}$, and a mapping $f : (X, \tau_X) \rightarrow (X, \tau_X)$ is defined as $f(a) = a$, $f(b) = c$ and $f(c) = b$, hence $f$ is gs-continuous and sg-continuous but not strongly $g^*s$-continuous.

**Theorem 4.1.** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is strongly $g^*s$-continuous iff the inverse image of each open set in $Y$ is strongly $g^*s$-open in $X$.

**Proof.** The necessity. Let $G \subseteq Y$ be an open set. Then, $Y - G$ is closed, hence, by hypothesis, $f^{-1}(Y - G)$ is a strongly $g^*s$-closed set. Therefore, $f^{-1}(G)$ is strongly $g^*s$-open.

The sufficiency. Let $F \subseteq Y$ be a closed set. Then, $Y - F$ is open, hence by hypothesis, $f^{-1}(Y - F)$ is a strongly $g^*s$-open set. Thus $f^{-1}(F)$ is strongly $g^*s$-closed. So, $f$ is strongly $g^*s$-continuous.

**Lemma 4.2.** Every strongly $g^*s$- irresolute mapping is strongly $g^*s$-continuous.

**Example 4.5.** Let $X = \{a,b,c\}$, $Y = \{a,b,c\}$, and a mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ be defined by $f(a) = f(b) = a$ and $f(c) = c$. Then, $f$ is strongly $g^*s$-continuous but not strongly $g^*s$- irresolute.

**Remark 4.3.** The composition of two strongly $g^*s$- continuous mappings may not be strongly $g^*s$- continuous the following example shows this fact.
Example 4.6. Let $X = \{a,b,c\}$ and $Y = \{a,b,c,d\}$ with the topologies $\tau_X = \{X, \emptyset, \{a\}\}$, $\tau_Y = \{Y, \emptyset, \{a,c\}\}$, $\tau_Z = \{Z, \emptyset, \{c\}\}$, a mapping $f$ from $(X, \tau_X)$ to $(Y, \tau_Y)$ is the identity map and a mapping $g : (Y, \tau_Y) \rightarrow (Z, \tau_Z)$ is defined by $g(a) = a$, $g(b) = g(d) = b$ and $g(c) = c$. Then, $f$ and $g$ are strongly $g$-continuous, but $g \circ f$ is not strongly $g^s$-continuous.

In the next theorem, we give the necessarily condition which satisfying the composition of two strongly $g^s$-continuous mappings is also strongly $g^s$-continuous.

**Theorem 4.2.** Let $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ and $g : (Y, \tau_Y) \rightarrow (Z, \tau_Z)$ be two mappings. Then, $g \circ f : (X, \tau_X) \rightarrow (Z, \tau_Z)$ is strongly $g^s$-continuous if one of the following conditions are satisfied.

(i) $f$ is strongly $g^s$-continuous and $g$ is continuous.

(ii) $f$ is semi-continuous and $g$ is continuous.

(iii) $f$ is strongly $g^s$-irresolute and $g$ is strongly $g^s$-continuous.

**Proof.**

(i) Let $F \subseteq Z$ be a closed set and $g$ be a continuous mapping. Then, $g^{-1}(F) \subseteq Y$ is closed. But, $f$ is strongly $g^s$-continuous, then $f^{-1}(g^{-1}(F)) \subseteq X$ is strongly $g^s$-closed. Therefore, $(g \circ f)^{-1}(F)$ is strongly $g^s$-closed in $X$.

(ii) If $V$ is a closed subset of $Z$, then $g^{-1}(V) \subseteq Y$ is closed. But, $f$ is semi-continuous, then $f$ is strongly $g^s$-continuous, hence $(g \circ f)^{-1}(V)$ is strongly $g^s$-closed in $X$.

(iii) Let $V$ be a closed subset of $Z$ and $g$ is strongly $g^s$-continuous. Then, $g^{-1}(V) \subseteq Y$ is strongly $g^s$-closed. But, $f$ is strongly $g^s$-irresolute, then $f^{-1}(g^{-1}(V)) \subseteq X$ is strongly $g^s$-closed. Hence, $g \circ f$ is strongly $g^s$-continuous.

5. Strongly $g^s$-closed mappings.

**Definition 5.1.** A mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called strongly generalized star semi-closed (resp. strongly generalized star semi-open) (briefly, strongly $g^s$-closed and strongly $g^s$-open) if the image of each closed (resp. open) set of $X$ is strongly $g^s$-closed (resp. strongly $g^s$-open) in $Y$.

**Remark 5.1.** The $g$-closed (resp. $g$-open) and strongly $g^s$-closed (resp. strongly $g^s$-open) mappings are independent. The following examples show this remark.

**Example 5.1.** Let $X = \{a,b,c,d\}$ and $\tau_X = \{X, \emptyset, \{a\}\}$, $\tau_Y = \{Y, \emptyset, \{a,c\}\}$, $\{a,b,c\}$ be two topologies on $X, Y$ respectively. Then, the mapping $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ which is defined by $f(a) = c$, $f(b) = a$, $f(c) = b$ and $f(d) = d$ is $g$-closed (resp. $g$-open) but not strongly $g^s$-closed (resp. strongly $g^s$-open).

**Example 5.2.** Let $X = \{a,b,c,d\}$ with two topologies $\tau_X = \{X, \emptyset, \{b,c,d\}\}$ and $\tau_Y = \{Y, \emptyset, \{a\}\}$, $\{b,c\}$, $\{a,b,c\}$). Then, the identity mapping from $(X, \tau_X)$ into $(Y, \tau_Y)$ is strongly $g^s$-closed (resp. strongly $g^s$-open) but not $g$-closed (resp. $g$-open).

**Remark 5.2.** It is clear that a strongly $g^s$-closed (resp. strongly $g^s$-open) mapping is weaker than semi-closed (resp. semi-open) and stronger than each of $sg$-closed (resp. $sg$-open). The implications between these new types of mappings and other corresponding ones are given by the following diagram.

```
+---------------------------+                      +---------------------------+
| closed                   |                      | closed                   |
| (open)                   |                      | (g-open)                 |
|                           |                      |                           |
| semi-closed              |                        | strongly $g^s$-closed    |
| (semi-open)              |                        | (strongly $g^s$-open)    |
|                           |                        |                           |
| g-closed                 |                        | sg-closed                |
| (g-open)                 |                        | (sg-open)                |
|                           |                        |                           |
|                           | g-closed               |                           |
|                           | (g-open)               |                           |
|                           |                         |                           |
|                           |                          | sg-closed                |
|                           |                          | (sg-open)                |
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The converses of these implications are not true in [11,16,18] and by the following examples.
Example 5.3. If \( X \), \( Y \) \{a, b, c, d\} and \( \tau_X \) \{X, \varphi, \{a\}, \{a, b\}\}, \( \tau_Y \) \{Y, \varphi, \{c, d\}\}, then a mapping \( f: (X, \tau_X) \to (Y, \tau_Y) \) which defined by \( f(a) = c \), \( f(b) = d \), \( f(c) = a \) and \( f(d) = b \) is strongly \( g^s \)-closed (resp. strongly \( g^s \)-open) but it is not semi-closed (resp. semi-open).

Example 5.4. If \( X \), \( Y \) \{a, b, c\} with two topologies \( \tau_X \) \{X, \varphi, \{a\}, \{a, b\}\} and \( \tau_Y \) \{Y, \varphi, \{a\}, \{b, c\}\}, then a mapping \( f: (X, \tau_X) \to (Y, \tau_Y) \) which is defined by \( f(a) = a \), \( f(b) = c \) and \( f(c) = b \) is gs-closed (resp. gs-open) and sg-closed (resp. sg-open) but not strongly gs-closed (resp. strongly gs-open).

**Theorem 5.1.** For a bijective mapping \( f: (X, \tau) \to (Y, \sigma) \), the following statements are equivalent:

(i) \( f \) is strongly \( g^s \)-closed,

(ii) \( f \) is strongly \( g^s \)-open,

(iii) \( f^{-1} \) is strongly \( g^s \)-continuous.

**Proof.** (i)\( \to \) (ii). Let \( G \subseteq X \) be an open set. Then, \( X - G \) is closed and by hypothesis, \( f(X - G) \) is strongly \( g^s \)-closed. Since, \( f \) is bijective, hence \( Y - f(G) \) is strongly \( g^s \)-closed. Therefore, \( f(G) \) is strongly \( g^s \)-open.

(ii)\( \to \) (iii). If \( G \subseteq X \) is an open set, then \( f(G) \) is strongly \( g^s \)-open in \( Y \). Since, \( f \) is bijective, hence \( (f^{-1})^{-1}(G) \) is strongly \( g^s \)-closed in \( Y \). Therefore, \( f^{-1} \) is strongly \( g^s \)-continuous.

(iii)\( \to \) (i). Let \( F \subseteq X \) be a closed set. Then, \( (f^{-1})^{-1}(F) \) is strongly \( g^s \)-closed in \( Y \). But, \( f \) is bijective, hence \( f(F) \) is strongly \( g^s \)-closed in \( Y \). So, \( f \) is strongly \( g^s \)-closed.

**Theorem 5.2.** A mapping \( f: (X, \tau) \to (Y, \sigma) \) is strongly \( g^s \)-open (resp. strongly \( g^s \)-closed) iff for any subset \( A \) in \( (Y, \sigma) \) and any closed (resp. open) set \( F \) in \( (X, \tau) \) containing \( f^{-1}(A) \), there exists a strongly \( g^s \)-closed (resp. strongly \( g^s \)-open) subset \( B \) of \( (Y, \sigma) \) containing \( A \) such that \( f^{-1}(B) \subseteq F \).

**Proof.** The necessity. Let \( f: (X, \tau) \to (Y, \sigma) \) be a strongly \( g^s \)-open mapping and \( F \) be a closed set containing \( f^{-1}(A) \) where \( A \subseteq Y \). Then, \( f(X - F) \) is strongly \( g^s \)-open in \( Y \). Set, \( Y - f(X - F) \). Since, \( f^{-1}(A) \subseteq F \), hence \( X - F \subseteq X - f^{-1}(A) \). Therefore, \( f(X - F) \subseteq Y - A \). Then, \( A \subseteq Y - f(X - F) \). Where, \( Y - f(X - F) \), then \( f^{-1}(B) \subseteq Y - f(X - F) \). Hence, \( f^{-1}(B) \subseteq F \).

The sufficiency. Let \( U \) be an open set in \( X \). Then, \( X - U \) is closed such that \( f^{-1}(Y - f(U)) \subseteq X - U \). By hypothesis, there exists a strongly \( g^s \)-closed set \( B \) containing \( Y - f(U) \), that is, \( Y - f(U) \subseteq B \). Also, since, \( f^{-1}(B) \subseteq X - U \), then \( f(U) \subseteq f(X - f^{-1}(B)) \subseteq Y - B \). This implies that \( B \subseteq Y - f(U) \). Hence, from (1),(2) we have \( B \subseteq Y - f(U) \) which is strongly \( g^s \)-closed. So, \( f(U) \) is strongly \( g^s \)-open. Therefore, \( f: (X, \tau) \to (Y, \sigma) \) is strongly \( g^s \)-open.

By similarly, we can prove this theorem for a case, if, \( f: (X, \tau) \to (Y, \sigma) \) is strongly \( g^s \)-closed.

**Remark 5.3.** The composition of two strongly \( g^s \)-closed (resp. strongly \( g^s \)-open) mappings may not be strongly \( g^s \)-closed (resp. strongly \( g^s \)-open). The following examples show this fact.

**Example 5.5.** Let \( X \), \( Y \), \( Z \) \{a, b, c, d\} with topologies \( \tau_X \) \{X, \varphi, \{a\}, \{a, b\}\} \{a, c, d\}\}, \( \tau_Y \) \{Y, \varphi, \{c, d\}\} and \( \tau_z \) \{Z, \varphi, \{c, d\}\}. Then, a mapping \( f: (X, \tau_X) \to (Y, \tau_Y) \) which defined by \( f(a) = a \), \( f(b) = d \), \( f(c) = b \) and \( f(d) = c \) and a mapping \( g: (Y, \tau_Y) \to (Z, \tau_z) \) which also defined by \( g(a) = g(b) = a \), \( g(c) = c \) and \( g(d) = b \) are strongly \( g^s \)-closed, but \( g \circ f \) is not strongly \( g^s \)-closed.

**Example 5.6.** Let \( X \), \( Y \), \( Z \) \{a, b, c, d\} with topologies \( \tau_X \) \{X, \varphi, \{a\}, \{a, b\}\} \{a, c, d\}\}, \( \tau_Y \) \{Y, \varphi, \{a\}, \{b, c\}\} \{a, b, c\}\}, \( \tau_z \) \{Z, \varphi, \{c, d\}\} and topologies \( \tau_z \) \{Z, \varphi, \{c, d\}\}. Then, a mapping \( f: (X, \tau_X) \to (Y, \tau_Y) \) which defined by \( f(a) = a \), \( f(b) = d \), \( f(c) = c \) and
\[ \begin{align*}
&f(d) \quad b \quad \text{and a mapping} \quad g: (Y, \tau_Y) \to (Z, \tau_Z) \quad \text{which also defined by} \quad g(a) \quad g(c) \quad c \quad g(b) \quad d \quad \text{and} \quad g(d) \quad b \\
&\text{are strongly} \quad g^{s} \quad \text{-open, but} \quad g \circ f \quad \text{is not strongly} \quad g^{s} \quad \text{-open.}
\end{align*} 

In the following, we give the conditions under which the composition of two strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open) may be strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open).

**Theorem 5.3.** Let \( f: (X, \tau_X) \to (Y, \tau_Y) \) and \( g: (Y, \tau_Y) \to (Z, \tau_Z) \) be two mappings. Then, the following statements are hold:

(i) If \( f \) is closed (resp. open) and \( g \) is strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open), then \( g \circ f \) is strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open).

(ii) If \( g \circ f \) is strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open) and \( f \) is surjective continuous, then \( g \) is strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open).

(iii) If \( g \circ f \) is closed (resp. open) and \( g \) is injective strongly \( g^{s} \)-continuous then, \( f \) is strongly \( g^{s} \)-closed (resp. strongly \( g^{s} \)-open).

**Proof.**

(i) Let \( G \) be a closed subset of \( X \). Then, \( f(G) \) is closed in \( Y \). But, \( g \) is strongly \( g^{s} \)-closed, then \( g(f(G)) \) is strongly \( g^{s} \)-closed in \( Z \). Therefore, \( g \circ f(G) \) is strongly \( g^{s} \)-closed.

(ii) If \( F \) is closed set in \( Y \), then \( f^{-1}(F) \) is closed in \( X \). Hence, by hypothesis, \( (g \circ f)(f^{-1}(F)) \) is strongly \( g^{s} \)-closed. Since, \( f \) is surjective, then \( g(F) \) is strongly \( g^{s} \)-closed. Therefore, \( g \) is strongly \( g^{s} \)-closed.

(iii) If \( F \) is closed set in \( X \), then \( g \circ f(F) \) is closed in \( Z \). Hence, by hypothesis, \( g^{-1}(g \circ f(F)) \) is strongly \( g^{s} \)-closed. Since, \( g \) is injective, then \( f(F) \) is strongly \( g^{s} \)-closed. Therefore, \( f \) is strongly \( g^{s} \)-closed.

6. **strongly \( g^{s} \)-homeomorphisms.**

**Definition 6.1.** A bijection \( f: (X, \tau_X) \to (Y, \sigma_Y) \) is called a strongly \( g^{s} \)-homeomorphism if \( f \) is both strongly \( g^{s} \)-continuous and strongly \( g^{s} \)-open.

**Remark 6.1.**

(1) Every semi-homeomorphism (B) is strongly \( g^{s} \)-homeomorphism.

(2) Every strongly \( g^{s} \)-homeomorphism is sg-homeomorphism (resp. gs-homeomorphism).

The converse of above remark is not true as is shown by the following examples.

**Example 6.1.** Let \( X = \{a, b, c, d\} \) with two topologies \( \tau_X = \{X, \varnothing, \{c, d\}\} \) and \( \tau_Y = \{Y, \varnothing, \{a\}, \{b\}, \{a, b\}\} \). Then, a mapping \( f: (X, \tau_X) \to (Y, \tau_Y) \) which defined by \( f(a) \quad d \quad f(b) \quad c \quad f(c) \quad a \quad \text{and} \quad f(d) \quad b \) is strongly \( g^{s} \)-homeomorphism but not semi-homeomorphism (B).

**Example 6.2.** If \( X = \{a, b, c\} \) with topology \( \tau_X = \{X, \varnothing, \{a, b\}, \{c\}\} \) and, then a mapping \( f: (X, \tau_X) \to (X, \tau_X) \) which defined by \( f(a) \quad a \quad f(b) \quad c \quad \text{and} \quad f(c) \quad b \) is sg-homeomorphism and gs-homeomorphism but not strongly \( g^{s} \)-homeomorphism.

By Remark 6.1 and the above examples we obtain the following diagram.

\[ \begin{array}{ccc}
\text{homeomorphism} & \longrightarrow & \text{semi-homeomorphism (B)} \\
\downarrow & & \downarrow \\
\text{strongly } g^{s} \text{-homeomorphism} & \quad & \quad \\
\downarrow & & \downarrow \\
\text{sg-homeomorphism} & \longrightarrow & \text{gs-homeomorphism}
\end{array} \]

**Proposition 6.1.** Let \( f: (X, \tau_X) \to (Y, \sigma_Y) \) be a bijective and strongly \( g^{s} \)-continuous map. Then, the following statements are equivalent:

(i) \( f \) is strongly \( g^{s} \)-open,
(ii) \( f \) is strongly \( gs \)-homeomorphism,

(iii) \( f \) is strongly \( gs \)-closed.

**Proof.** (i) \( \rightarrow \) (ii). It is clear from Definition 6.1.

(ii) \( \rightarrow \) (iii). Since, \( f \) is strongly \( gs \)-homeomorphism, then \( f \) is strongly \( gs \)-open. But, \( f \) is bijective, hence by Theorem 5.1, \( f \) is strongly \( gs \)-closed.

(iii) \( \rightarrow \) (i). Obvious.

**Remark 6.2.** The composition of two strongly \( gs \)-homeomorphism mappings may not be strongly \( gs \)-homeomorphism. The following example shows this fact.

**Example 6.3.** Let \( X, Y, Z \) \{a,b,c\} with topologies \( \tau_X \) \{X,\( \varphi \),\{a\}\}, \( \tau_Y \) \{Y,\( \varphi \),\{a,c\}\} and \( \tau_Z \) \{Z,\( \varphi \),\{c\}\}. Then, a mapping \( f: (X, \tau_X) \rightarrow (Y, \tau_Y) \) which defined by \( f(a) \) \( b \), \( f(b) \) \( b \) and \( f(c) \) \( a \) and a mapping \( g: (Y, \tau_Y) \rightarrow (Z, \tau_Z) \) which also defined by \( g(a) \) \( c \), \( g(b) \) \( b \) and \( g(c) \) \( a \) are strongly \( gs \)-homeomorphism, but \( g \circ f \) is not strongly \( gs \)-homeomorphism.

**References**


