COUPLED COINCIDENCE POINT THEOREM FOR NONLINEAR
CONTRACTION IN PARTIALLY ORDERED METRIC SPACES

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INTRODUCTION

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions [1, 6, 7, 13]. Recently Agarwal [1], Bhaskar and Laxmikantham [4], Hussain [11], Nietz and Rodriguez-Lobez [13] presented some new results in partially ordered metric spaces. D. Guo and Laxmikantham [9, 10] introduced the concept of mixed monotone operator and the coupled fixed points. Later on several authors [5, 8, 11, 14, 15] have used this concept and proved the existence of coupled fixed points for mixed monotone operators. Ciric and Laxmikantham [8] generalized the concept of mixed monotone to mixed g-monotone and have obtained existence theorems for coupled fixed points.

In this paper, the existence theorem of coupled coincidence point is proved. The main tool in the proof of result combines the ideas in the contraction principle with those in the monotonic iterative technique. An example is given satisfying contractive type condition.

Abstract: The existence theorem of coupled coincidence point is proved. The main tool in the proof of result combines the ideas in the contraction principle with those in the monotonic iterative technique. An example is given satisfying contractive type condition.
Definition 1.4 A function \( \psi : [0, \infty) \rightarrow [0, \infty) \) is said to be convex if 
\[
\psi(at_1 + (1 - a)t_2) \leq a \psi(t_1) + (1 - a) \psi(t_2)
\]
where \( t_1, t_2 \in [0, \infty) \) and \( a \in [0, 1] \).

Definition 1.5 A function \( \psi : [0, \infty) \rightarrow [0, \infty) \) is said to be affine if 
\[
\psi(at_1 + (1 - a)t_2) = a \psi(t_1) + (1 - a) \psi(t_2)
\]
where \( t_1, t_2 \in [0, \infty) \) and \( a \in [0, 1] \).

Altering functions have been used in metric fixed point theory in recent papers [6, 2, 12, 3].

2 Main Results

Analogous with definition 1.1, Laxmikantham and Ciric [8] introduced following concept of mixed \( g \)-monotone mapping.

Definition 2.1[8] Let \((X, \leq)\) be a partially ordered set and \( f : X \times X \rightarrow X \) and \( g : X \times X \rightarrow X \). We say that mapping \( f \) has mixed \( g \)-monotone property if \( f \) is nondecreasing \( g \)-monotone in its first argument and is nonincreasing \( g \)-monotone in its second argument, i.e. for any \( x, y \in X \),
\[
x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow f(x_1, y) \leq f(x_2, y);
\]
\[
y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow f(x, y_1) \leq f(x, y_2).
\]
Note that if \( g \) is identity mapping, then definition 2.1 reduces to definition 1.1.

Definition 2.2[8] An element \((x, y) \in X \times X\) is called a coupled coincidence point of the mappings \( f : X \times X \rightarrow X \) and \( g : X \times X \rightarrow X \) if
\[
g(f(x, y)) = f(gx, gy),
\]
for all \( x, y \in X \).

Definition 2.3[8] Let \( X \) be a nonempty set and \( f : X \times X \rightarrow X \) and \( g : X \times X \rightarrow X \). and \( g \) are said to have commutative property if
\[
g(f(x, y)) = f(gx, gy),
\]
for all \( x, y \in X \).

Suppose \((X, d)\) be a metric space and \( f : X \times X \rightarrow X \), \( g : X \times X \rightarrow X \) such that \( f(X \times X) \subseteq g(X) \). For any \( x_0, y_0 \in X \), we can choose \( x_1, y_1 \in X \) such that \( g(x_1) = f(x_0, y_0) \) and \( g(y_1) = f(y_0, x_0) \). Similarly for \( x_2, y_2 \in X \) such that \( g(x_2) = f(x_1, y_1) \) and \( g(y_2) = f(y_1, x_1) \). Continuing this process we can construct sequences \([gx_n]\) and \([gy_n]\) in \( X \) such that
\[
\begin{align*}
gx_{n+1} &= f(x_n, y_n) \quad \text{and} \quad gy_{n+1} = f(y_n, x_n). \quad (2.1)
\end{align*}
\]
For the main result we need following assumptions.

\( (H_1) \) \( f(X \times X) \subseteq g(X) \).

\( (H_2) \) \( f \) has mixed \( g \)-monotone property.

\( (H_3) \) For any \( x, y, u, v \in X \),
\[
\psi(d(f(x, y), f(u, v))) \leq \psi(d(gx, gu) + d(gy, gv)) \psi(\frac{d(gx, gu) + d(gy, gv)}{2})
\]
where \( \beta \in S \) and \( \psi \) is convex altering function.

Theorem 2.1 Let \((X, d)\) be a metric space and \( f : X \times X \rightarrow X \), \( g : X \times X \rightarrow X \). Suppose \( (H_3) \) holds. If \((x, y) \in X \times X\) is a coupled coincidence of \( g \) and \( f \) then \( gx = gy \). Moreover, if \((x, y) \) and \((x_0, y_0)\) are coupled coincidences of \( g \) and \( f \) then \( gx = gx_0 = gy = gy_0 \).

Proof: Suppose \((x, y) \in X \times X\) is a coupled coincidence of \( g \) and \( f \). Therefore,
\[
g(x) = f(x, y); \quad g(y) = f(y, x).
\]
From \( (H_3) \), we get
\[
\psi(d((gx, gy))) = \psi(d(f(x, y), f(y, x))) \leq \beta(d(gx, gy) + d(gy, gx)) \psi\left(\frac{d(gx, gy) + d(gy, gx)}{2}\right) = \beta(2d(gx, gy)) \psi(dgx, gy)).
\]
Since \( \psi \) is an altering function and \( \beta \in S \), from the above inequality we get \( d(gx, gy) = 0 \) which implies \( gx = gy \).

Now suppose \((x, y)\) and \((x', y')\) are coupled coincidences of \( g \) and \( f \). Therefore,
\[
g(x) = f(x, y); \quad g(y) = f(y, x).
\]
and 

\[ g(x') = f(x', y'); g(y') = f(y', x'). \]

Moreover we have \( gx = gy \) and \( gx' = gy' \). From \((H_i)\) we get,

\[
\psi(d(gx, gx')) = \psi(d(f(x, y), f(x', y'))) \\
\leq \beta(d(gx, gx') + d(gy, gy'))\psi\left(\frac{d(gx, gx') + d(gy, gy')}{2}\right) \\
= \beta(d(gx, gx') + d(gx, gx'))\psi\left(\frac{d(gx, gx') + d(gx, gx')}{2}\right) \\
= \beta(2d(gx, gx'))\psi(d(gx, gx')).
\]

Since \( \psi \) is an altering function and \( \beta \in S \), from the above inequality we get \( d(gx, gx') = 0 \) which implies \( gx = gx' \).

This completes the proof.

For the main result we need the following lemma.

**Lemma 2.1** Let \( (X, \leq, d) \) be a partially ordered metric space and \( f : X \times X \rightarrow X, g : X \rightarrow X \). Assume \((H_1) - (H_3)\) hold. Suppose there exists \( x_0, y_0 \in X \) such that

\[ gx_0 \leq f(x_0, y_0) \text{ and } f(y_0, x_0) \leq gy_0 \] \hspace{1cm} (2.2)

If the sequences \( \{gx_n\}, \{gy_n\} \) are defined by (2.1) then

(a) \( \{gx_n\} \) is nondecreasing and \( \{gy_n\} \) is nonincreasing sequence.

(b) \( \lim_{n \to \infty} d(gx_n, gx_{n+1}) = 0 \) \hspace{1cm} (2.3)

(c) \( \lim_{n \to \infty} d(gy_n, gy_{n+1}) = 0 \) \hspace{1cm} (2.4)

Proof: Hypothesis \((H_1)\) implies that the sequences \( \{gx_n\} \) and \( \{gy_n\} \) defined by (2.1) exist.

(a) To prove

\[ gx_{n+1} \geq gx_n \] \hspace{1cm} (2.5)

and

\[ gy_{n+1} \leq gy_n \] \hspace{1cm} (2.6)

we use mathematical induction. By (2.2), it is obvious that \( gx_0 \leq gx_1 \) and \( gy_0 \geq gy_1 \). Thus (2.5) and (2.6) hold for \( n = 0 \). Suppose now that (2.5) and (2.6) hold for some fixed \( n \geq 0 \). Then, since \( gx_n \leq gx_{n+1} \) and \( gy_n \geq gy_{n+1} \) and as \( f \) has \( g \)-mixed monotone property,

\[ gx_{n+1} = f(x_n, y_n) \leq f(x_{n+1}, y_n) \leq f(x_{n+1}, y_{n+1}) = gx_{n+2}, \]

\[ gy_{n+1} = f(y_n, x_n) \geq f(y_{n+1}, x_n) \geq f(y_{n+1}, x_{n+1}) = gy_{n+2}. \]

Thus by the mathematical induction, we conclude that (2.5) and (2.6) hold for all \( n \geq 0 \). Therefore,

\[ gx_0 \leq gx_1 \leq gx_2 \leq \ldots \leq gx_n \leq gx_{n+1} \leq \ldots \] \hspace{1cm} (2.7)

\[ gy_0 \geq gy_1 \geq gy_2 \geq \ldots \geq gy_n \geq gy_{n+1} \geq \ldots \] \hspace{1cm} (2.8)

To prove (b), denote \( d_n = d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1}) \). Using \((H_3)\) we obtain

\[
\psi\left(d(gx_n, gx_{n+1})\right) = \psi\left(d(f(x_{n-1}, y_{n-1}), f(x_n, y_n))\right) \\
\leq \beta\left(d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)\right)\psi\left(\frac{d(gx_{n-1}, gx_n) + d(gy_{n-1}, gy_n)}{2}\right)
\]
\[
\leq \beta(\delta_{n-1}) \psi\left(\frac{(\delta_{n-1})}{2}\right)
\]

Similarly we obtain,
\[
\psi(d(gy_n, gy_{n+1})) \leq \beta(\delta_{n-1}) \psi\left(\frac{(\delta_{n-1})}{2}\right)
\]

Since \(\psi\) is a convex function,
\[
\begin{align*}
\psi\left(\frac{\delta_n}{2}\right) &= \psi\left(\frac{d(g(x_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2}\right) \\
&\leq \frac{1}{2} \psi(d(gx_n, gy_n)) + \frac{1}{2} \psi(d(gy_n, gy_{n+1})) \\
&\leq \frac{1}{2} \beta(\delta_{n-1}) \psi\left(\frac{(\delta_{n-1})}{2}\right) + \frac{1}{2} \beta(\delta_{n-1}) \psi\left(\frac{(\delta_{n-1})}{2}\right) \\
&= \beta(\delta_{n-1}) \psi\left(\frac{\delta_{n-1}}{2}\right).
\end{align*}
\] (2.9)

Moreover, \(\beta\) is nondecreasing and \(\beta \in S\), hence we have \(\delta_{n-1}/2 \leq \delta_{n-1}/2\)

Therefore,
\[
\delta_n \leq \delta_{n-1}
\] (2.10)

If there exists \(n_0\) such that \(\delta_{n_0} = 0\) then obviously (2.3), (2.4) hold.

In other case, suppose \(\delta_n \neq 0\) for all \(\beta \in N\). Then taking into account (2.10), the sequence \(\{\delta_n\}\) is decreasing and bounded below. So
\[
\lim_{n \to \infty} \delta_n = r \geq 0
\] (2.11)

Assume \(r > 0\), then from (2.9), we have
\[
\begin{align*}
\frac{\psi\left(\frac{\delta_n}{2}\right)}{\psi\left(\frac{\delta_{n-1}}{2}\right)} \leq \beta(\delta_{n-1}) < 1.
\end{align*}
\] (2.12)

Letting \(n \to 1\) in the last inequality and by the fact that \(\beta\) is continuous, we get \(1 \leq \beta(\delta_{n-1}) < 1\).

Therefore
\[
\lim_{n \to \infty} \beta(\delta_n) = 1
\]

Since \(\beta \in S\), \(\lim_{n \to \infty} \beta(\delta_n) = 0\) and this contradicts to our assumption that \(r > 0\). Therefore \(r = 0\) and hence (2.3) and (2.4) hold.

To prove (c), it is sufficient to prove following two statements.

(i) At least one of sequences \(\{gx_n\}\) and \(\{gy_n\}\) is a Cauchy sequence.
(ii) If one of sequences \(\{gx_n\}\) and \(\{gy_n\}\) is a Cauchy sequence then so is other.

If possible suppose that \(\{gx_n\}\) and \(\{gy_n\}\) both are not Cauchy sequences. Therefore there exists \(\varepsilon_1 > 0\) and \(\varepsilon_2 > 0\) for which we can find subsequences \(\{gx_{n(k1)}\}\) and \(\{gy_{n(k2)}\}\) such that
\[ n(k_i) > m(k_i) > k_i; \ d(G_{n(k_i)}, G_{m(k_i)}) \geq \varepsilon_1 \]

and

\[ n(k_2) > m(k_2) > k_2; \ d(G_{n(k_2)}, G_{m(k_2)}) \geq \varepsilon_2. \]

Suppose \( \varepsilon = \min \{ \varepsilon_1, \varepsilon_2 \} \); \( k = \max \{ k_1, k_2 \} \). then for \( n(k) > m(k) \geq k, \)

\[ d(G_{n(k)}, G_{m(k)}) \geq \varepsilon \text{ and } d(G_{n(k)}, G_{m(k)}) \geq \varepsilon. \]

Corresponding to \( m(k) \) we can choose smallest \( n_1(k) \) and \( n_2(k) \) such that

\( n_1(k) > m(k), n_2(k) > m(k) \) satisfying \( d(G_{n_1(k)}, G_{m(k)}) \geq \varepsilon \); \( d(G_{n_2(k)}, G_{m(k)}) \geq \varepsilon \)

and

\[ d(G_{n_1(k) - 1}, G_{m(k)}) < \varepsilon \]

(2.13)

Using (2.13) and the triangular inequality, we have

\[ \varepsilon \leq d(G_{n_1(k)}, G_{m(k)}) \leq d(G_{n_1(k) - 1}, G_{m(k)}) + \varepsilon. \]

(2.14)

Letting \( n \to 1 \) and using (2.3) we get

\[ \varepsilon \leq \lim_{k \to \infty} d(G_{n_1(k)}, G_{m(k)}) \leq \varepsilon. \]

Therefore

\[ \lim_{k \to \infty} d(G_{n_1(k)}, G_{m(k)}) = \varepsilon. \]

(2.15)

Similarly we obtain

\[ \lim_{k \to \infty} d(G_{n_2(k)}, G_{m(k)}) = \varepsilon. \]

(2.16)

Again, the triangular inequality gives

\[ d(G_{n_1(k)}, G_{m(k)}) \leq d(G_{n_1(k)}, G_{n_1(k) - 1}) + d(G_{n_1(k) - 1}, G_{m(k) - 1}) + d(G_{m(k) - 1}, G_{m(k)}). \]

and

\[ d(G_{n_1(k) - 1}, G_{m(k) - 1}) \leq d(G_{n_1(k) - 1}, G_{n_1(k)}) + d(G_{n_1(k)}, G_{m(k)}) + d(G_{m(k)}, G_{m(k) - 1}). \]

Letting \( k \to \infty \) in above inequality and using (2.3) and (2.15), we have

\[ \varepsilon \leq 0 + \lim_{k \to \infty} d(G_{n(k) - 1}, G_{m(k) - 1}) + 0. \]

And

\[ \lim_{k \to \infty} d(G_{n_1(k) - 1}, G_{m(k) - 1}) \leq 0 + \varepsilon + 0. \]

Therefore,

\[ \lim_{k \to \infty} d(G_{n_1(k) - 1}, G_{m(k) - 1}) = \varepsilon. \]

(2.17)

Similarly we obtain

\[ \lim_{k \to \infty} d(G_{n_2(k) - 1}, G_{m(k) - 1}) = \varepsilon. \]

(2.18)

Suppose \( n_2(k) > n_1(k) \), since \( m(k) \) is nondecreasing and using \((H_2), (2.7) \) and \((2.8) \) we have

\[ \psi(d(G_{n_1(k)}, G_{m(k)})) = d(f(x_{n_1(k) - 1}, y_{n_1(k) - 1}), f(x_{m_1(k) - 1}, y_{m_1(k) - 1})) \]

\[ \leq \beta(d(G_{n_1(k) - 1}, G_{m_1(k) - 1}) + d(G_{n_1(k)}, G_{m_1(k)})) \]
\[
\psi \left( \frac{d(g_{x_{n(k)} - 1}, g_{x_{m(k)} - 1}) + d(g_{y_{n(k)} - 1}, g_{y_{m(k)} - 1})}{2} \right)
\leq \beta(d(g_{x_{n(k)} - 1}, g_{x_{m(k)} - 1}) + d(g_{y_{n(k)} - 1}, g_{y_{m(k)} - 1}))
\]

Taking into account (2.15), (2.17), (2.18) and the fact that \( \psi \) is continuous, letting \( k \to \infty \) in (2.19), we get

\[
\psi(\varepsilon) \leq \beta (d(g_{x_{n(k)} - 1}, g_{x_{m(k)} - 1}) + d(g_{y_{n(k)} - 1}, g_{y_{m(k)} - 1})) \leq \psi(\varepsilon).
\]

As \( \psi \) is altering function, \( \psi(\varepsilon) > 0 \), the last inequality gives us

\[
\lim_{k \to \infty} \beta(d(g_{x_{n(k)} - 1}, g_{x_{m(k)} - 1}) + d(g_{y_{n(k)} - 1}, g_{y_{m(k)} - 1})) = 1.
\]

Since \( \beta \in S \), this means that

\[
\lim_{k \to \infty} d(g_{x_{n(k)} - 1}, g_{x_{m(k)} - 1}) = 0
\]

and

\[
\lim_{k \to \infty} d(g_{y_{n(k)} - 1}, g_{y_{m(k)} - 1}) = 0
\]

From (2.17) and (2.20) we get \( \varepsilon = 0 \) which is a contradiction.

If \( n_1(k) > n_2(k) \) then considering \( (d(g_{y_{n_2(k)}}, g_{y_{m(k)}})) \) and adopting same procedure as above, we get

\[
\lim_{k \to \infty} d(g_{x_{n_2(k)} - 1}, g_{x_{m(k)} - 1}) = 0
\]

and

\[
\lim_{k \to \infty} d(g_{y_{n_2(k)} - 1}, g_{y_{m(k)} - 1}) = 0
\]

From (2.18) and (2.21) we get \( \varepsilon = 0 \) which is again a contradiction. Therefore, at least one of the sequences \( \{g_m\} \) and \( \{g_m\} \) must be a Cauchy sequence. This proves statement (i).

To prove statement (ii), assume \( \{g_m\} \) is Cauchy sequence. If \( \{g_m\} \) is not a Cauchy sequence, then there exist \( \varepsilon > 0 \) for which we can find subsequence \( \{g_{n(k)}\} \) such that

\[
d(g_{x_{n(k)}}, g_{m(k)}) \geq \varepsilon
\]

for \( n(k) > m(k) > k \). Corresponding to \( m(k) \) choose \( n(k) \) in such a way that it is smallest integer with \( n(k) > m(k) \) and satisfying (2.22). Then

\[
d(g_{x_{n(k)} - 1}, g_{m(k)}) < \varepsilon.
\]

By similar procedure as adopted earlier, we obtain

\[
\lim_{k \to \infty} d(g_{x_{n(k)}}, g_{m(k)}) = \varepsilon
\]

and

\[
\lim_{k \to \infty} d(g_{m(k) - 1}, g_{m(k) - 1}) = \varepsilon
\]

Now using (H3) we have

\[
\psi(d(g_{x_{n(k)} - 1}, g_{m(k)})) = \psi(d(f(x_{n_{1(k)} - 1}, y_{n_{1(k)} - 1}), f(x_{m_{1(k)} - 1}, y_{m_{1(k)} - 1})))
\]
\[ \leq \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1})) \psi \left( \frac{d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1})}{2} \right). \]

(2.26)

Since \( \{gy_n\} \) is a Cauchy sequence, we have
\[ \lim_{k \to \infty} d(gy_{n(k)-1}, gy_{m(k)-1}) = 0. \] (2.27)

Now \( \psi \) is continuous and nondecreasing and using (2.24), (2.25), (2.27); taking limit \( k \to \infty \) in (2.26) we obtain
\[ \psi(\varepsilon) \leq \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1})) \psi \left( \frac{\varepsilon + 0}{2} \right) \leq \lim_{k \to \infty} \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1})) \psi(\varepsilon). \]

Since \( \psi \) is altering function, \( \psi(\varepsilon) > 0 \), therefore,
\[ \lim_{k \to \infty} \beta(d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1})) = 1. \]

Since \( \beta \in S \), we have
\[ \lim_{k \to \infty} d(gx_{n(k)-1}, gx_{m(k)-1}) = 0. \] (2.28)

and
\[ \lim_{k \to \infty} d(gy_{n(k)-1}, gy_{m(k)-1}) = 0. \]

Result (2.28) contradicts to (2.25). Hence sequence \( \{gx_n\} \) is a Cauchy sequence. Similarly we can prove that if \( \{gx_n\} \) is a Cauchy sequence then so is \( \{gy_n\} \).

Hence sequences \( \{gx_n\} \) and \( \{gy_n\} \) both are Cauchy sequences. Thus statements (i) and (ii) are satisfied. This proves (c).

**Remark 2.1:** The condition of convex on \( \psi \) in Lemma 2.1 may be replaced by affine.

**Theorem 2.2** Let \((X, \leq, d)\) be a partially ordered complete metric space and \(f: X \times X \to X\), \(g: X \to X\) satisfy \((H_1)-(H_3)\). Assume further that

(i) \(g\) and \(f\) are continuous,

(ii) \(g\) commutes with \(f\).

If there exists \(x_0, y_0 \in X\) such that
\[ gx_0 \leq f(x_0, y_0) \text{ and } f(y_0, x_0) \leq gy_0, \]
then \(f\) and \(g\) have a coupled coincidence, that is, there exists \(x, y \in X\) such that \(gx = f(x, y)\) and \(gy = f(y, x)\).

**Proof:** Choose \(x_0, y_0 \in X\) such that \(gx_0 \leq f(x_0, y_0)\) and \(f(y_0, x_0) \leq gy_0\).

Since \(f(X \times X) \subseteq g(X)\), we can constructed sequences \( \{gx_n\} \) and \( \{gy_n\} \) defined by (2.1). Using Lemma 2.1, we have
\[ \lim_{n \to \infty} d(gx_n, gx_{n+1}) = 0 \]
and
\[ \lim_{n \to \infty} d(gy_n, gy_{n+1}) = 0. \]

Suppose there exists an integer \(n_0\) such that
\[ d(gx_{n_0}, gx_{n_0+1}) = 0 \] (2.29)
and
\[ d(gy_{n_0}, gy_{n_0+1}) = 0. \] (2.30)

Since \(g\) commutes with \(f\), we have...
\[ g(gx_{n0}) = g(gx_{n0+1}) = g(f(x_{n0}, y_{n0})) = f(gx_{n0}, gy_{n0}). \]

and

\[ g(gx_{n0}) = f(gx_{n0}, gx_{n0}). \]

Therefore, \((gx_{n0}, gy_{n0})\) is a point of coupled coincidence. In another case, suppose there does not exist any \(n_0\) satisfying (3.29) and (3.30). From Lemma 2.1, we observe that sequences \([gx_n]\) and \([gy_n]\) defined by (2.1) are Cauchy sequences. Since \(X\) is complete, there exists \(x, y \in X\) such that

\[ \lim_{n \to \infty} gx_n = x, \quad \text{and} \quad \lim_{n \to \infty} gy_n = y. \]

The continuity of \(g\) yields

\[ \lim_{n \to \infty} g(gx_n) = gx, \quad \text{and} \quad \lim_{n \to \infty} g(gy_n) = gy. \]

Since \(g\) commutes with \(f\) we have

\[ g(gx_{n1}) = g(f(x_n, y_n)) = f(gx_n, gy_n) \quad (2.31) \]

and

\[ g(gy_{n1}) = g(f(y_n, x_n)) = f(gy_n, gx_n) \quad (2.32) \]

Taking limit as \(n \to \infty\) in (2.31) and (2.32) and using the continuity of \(f\), we get

\[ gx = \lim_{n \to \infty} g(gx_{n+1}) = \lim_{n \to \infty} g(f(x_n, y_n)) = \lim_{n \to \infty} f(gx_n, gy_n) = f(x, y) \]

and

\[ gy = \lim_{n \to \infty} g(gy_{n+1}) = \lim_{n \to \infty} g(f(y_n, x_n)) = \lim_{n \to \infty} f(gy_n, gx_n) = f(y, x). \]

Thus \(gx = f(x, y)\) and \(gy = f(y, x)\). Therefore, \(g\) and \(f\) have a coupled coincidence.

**Corollary 2.1** Let \((X, \leq, d)\) be a partially ordered complete metric space and \(f : X \times X \to X\) be continuous mixed monotone mapping satisfying

\[ \psi(d(f(x, y), f(u, v))) \leq \beta(d(x, u) + d(y, v)) \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \]

where \(\beta \in S\) and \(\psi\) is convex (or affine) altering function. If there exists \(x_0, y_0 \in X\) such that

\[ x_0 \leq f(x_0, y_0) \quad \text{and} \quad f(x_0, y_0) \leq y_0 \]

then \(f\) has a unique fixed point.

**Proof:** Taking \(gx = x\), in Theorem 2.2 and using Theorem 2.1, we obtain Corollary 2.1.

**Corollary 2.2** Let \((X, \leq, d)\) be a partially ordered complete metric space and \(f : X \times X \to X\) be continuous mixed monotone mapping satisfying

\[ d(f(x, y), f(u, v)) \leq \beta(d(x, u) + d(y, v)) \psi\left(\frac{d(x, u) + d(y, v)}{2}\right) \]

where \(\beta \in S\). If there exists \(x_0, y_0 \in X\) such that

\[ x_0 \leq f(x_0, y_0) \quad \text{and} \quad f(y_0, x_0) \leq y_0 \]

then \(f\) has a unique coupled fixed point.

**Proof:** Letting \(\psi(t) = t\), in Corollary 2.1, we obtain Corollary 2.2.

The following example illustrates Corollary 2.2.

**Example 2.1** Let \(X = [1 - a, a]\) with usual partial order \(\leq\) and usual metric \(d\). Clearly \((X, \leq, d)\) is a partially ordered complete metric space. Let \(\psi(t) = t\) and \(\beta = 2a/2a + a\), obviously \(\psi\) is convex (affine) altering function and \(\beta \in S\). Define \(f : X \times X \to X\) by
Then \( f \) is continuous and has mixed monotone property. For \( x, y, u, v \in X \), we have

\[
d(f(x, y), f(u, v)) = d\left(\frac{x - y}{k}, \frac{u - v}{k}\right)
\]

\[
= \left| \frac{x - y}{k} - \frac{u - v}{k} \right|
\]

\[
\leq \frac{1}{k} |x - u| + \frac{1}{k} |y - v|.
\]

\[
d(x, u) + d(y, v) = |x - u| + |y - v|.
\]

\[
\beta d(x, u) + d(y, v) = \frac{2a}{2a + d(x, u) + d(y, v)} \geq \frac{2a}{2a + 2a + 2a} = \frac{1}{3}
\]

Therefore

\[
\beta d((x, u) + d(u, v)) \left(\frac{d(x, u) + d(y, v)}{2}\right) \geq \frac{1}{3} \cdot \frac{1}{3} (|x - u| + |y - v|)
\]

\[
\geq \frac{1}{k} (|x - u| + |y - v|)
\]

\[
\geq d(f(x, y), f(u, v)).
\]

Since \( k \geq 6 \), we can choose infinitely many \( x_0, y_0 \in X \) satisfying,

\[
(k - 1) x_0 + y_0 \leq 0 \quad \text{and} \quad x_0 + (k - 1) y_0 \geq 0
\]

(2.23).

This implies that

\[
x_0 \leq \frac{x_0 - y_0}{k} \quad \text{and} \quad y_0 - x_0 \leq y_0.
\]

Therefore,

\[
x_0 \leq f(x_0, y_0) \quad \text{and} \quad f(y_0, x_0) \leq y_0.
\]

Hence, by Corollary 2.2, \( f \) has a unique coupled fixed point.

**Remark 2.2** In example 2.1, \( x = 0 \) is the unique fixed point of \( f \).

**Remark 2.3** For any \( x_0, y_0 \in X \) satisfying (2.23), define sequences \( \{x_n\}, \{y_n\} \) by

\[
x_{n+1} = f(x_n, y_n), \quad y_{n+1} = f(y_n, x_n).
\]

Clearly we get

\[
x_{n+1} = \left(\frac{2}{k}\right)^n y_1; \quad y_{n+1} = \left(\frac{2}{k}\right)^n y_1.
\]

Since \( k \geq 6 \), \( x_n \to 0 \) and \( y_n \to 0 \) as \( n \to \infty \). Therefore, sequences \( \{x_n\}, \{y_n\} \) converges to a fixed point of \( f \).

**REFERENCES**


